Graduate Seminar in Geometry

Summersemester 2023

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Minimal Submanifolds

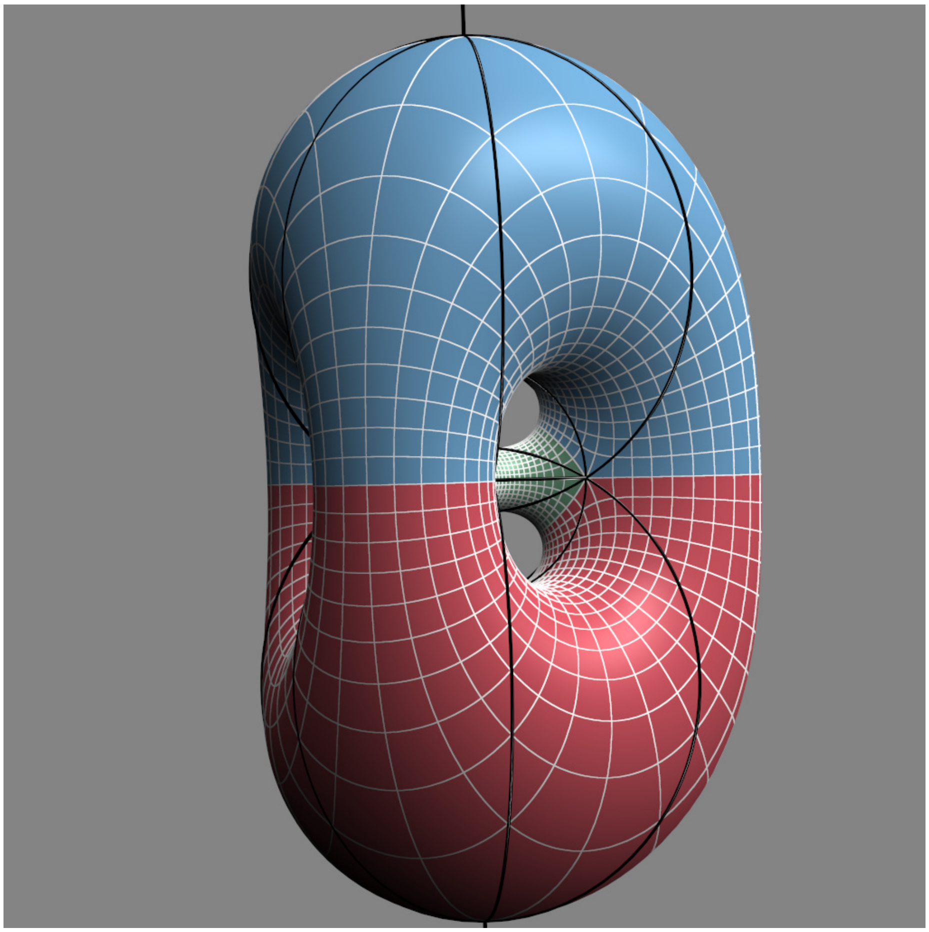

Figure 1: *Lawson's minimal surface of genus 4 inside the 3-dim'l sphere.*

A minimal surface is a surface that locally minimizes its area. This is equivalent to having zero mean curvature. They are the 2-dimensional analogue to geodesics, which are analogously defined as critical points of the length functional.

Minimal surface theory originates with Lagrange who in 1762 considered the variational problem of finding the surface $z = z(x, y)$ of least area stretched across a given closed contour. He derived the Euler–Lagrange equation. He did not succeed in finding any solution beyond the plane. In 1776 Jean Baptiste Marie Meusnier discovered that the helicoid and catenoid satisfy the equation and that the differential expression corresponds to twice the mean curvature of the surface, concluding that surfaces with zero mean curvature are area-minimizing. By expanding Lagrange's equation, Gaspard Monge and Legendre in 1795 derived representation formulas for the solution surfaces. While these were successfully used by Heinrich Scherk in 1830 to derive his surfaces, they were generally regarded as practically unusable. Catalan proved in 1842/43 that the helicoid is the only ruled minimal surface. Progress had been fairly slow until the middle of the century when the Björling problem was solved using complex methods. The "first golden age" of minimal surfaces began. Schwarz found the solution of the Plateau problem for a regular quadrilateral in 1865 and for a general quadrilateral in 1867 using complex methods. Weierstrass and Enneper developed more useful representation formulas, firmly linking minimal surfaces to complex analysis and harmonic functions. Other important contributions came from Beltrami, Bonnet, Darboux, Lie, Riemann, Serret and Weingarten.

Between 1925 and 1950 minimal surface theory revived, now mainly aimed at nonparametric minimal surfaces. The complete solution of the Plateau problem by Jesse Douglas and Tibor Radó was a major milestone. Bernstein's problem and Robert Osserman's work on complete minimal surfaces of finite total curvature were also important.Another revival began in the 1980s. One cause was the discovery in 1982 by Celso Costa of a surface that disproved the conjecture that the plane, the catenoid, and the helicoid are the only complete embedded minimal surfaces in R^3 of finite topological type. This not only stimulated new work on using the old parametric methods, but also demonstrated the importance of computer graphics to visualise the studied surfaces and numerical methods to solve the "period problem" (when using the conjugate surface method to determine surface patches that can be assembled into a larger symmetric surface, certain parameters need to be numerically matched to produce an embedded surface). Another cause was the verification by H. Karcher that the triply periodic minimal surfaces originally described empirically by Alan Schoen in 1970 actually exist. This has led to a rich menagerie of surface families and methods of deriving new surfaces from old, for example by adding handles or distorting them.

Currently the theory of minimal surfaces has diversified to minimal submanifolds in other ambient geometries, becoming relevant to mathematical physics (e.g. the positive mass conjecture, the Penrose conjecture) and three-manifold geometry (e.g. the Smith conjecture, the Poincaré conjecture, the Thurston Geometrization Conjecture).

Figure: Blaine Lawson's minimal surface $\xi_{2,2}$ of genus 4 inside the 3-dimensional sphere[Law70], which became a CMC (constant mean curvature) surface after the stereographic projection. *From:* Geometrie Werkstatt Gallery. TU-Berlin.

Prerequisites: Basic undergraduate Topology and Geometry.

Organizatorial Meeting: February 15 2023, 14:00, room 1.008

Talks:

1. Examples of minimal surfaces. *Ref:* [CM11] Section 1.2.
2. First variational formula for the volume functional. Mean curvature vector field on a Riemannian submanifold. *Ref:* [Li12] Chapter 1.
3. Second variation of energy for a minimally immersed submanifold. Stability of minimal submanifolds. *Ref:* [Li12] Chapter 1.
4. Higher osculating spaces. Higher dimensional curvatures and higher dimensional fundamental forms of a submanifold.
   Ref: [Eji83] Section 1 and [Che70] Section 2.
5. Minimal immersions into higher spheres: Frenet-Boruvka formulas for the minimal immersion of a 2-sphere in a constant curvature space.
   Ref: [Eji83] Sections 2. See also [Che70].
6. A Hermitian structure on the normal bundle. Jacobi operator, index and nullity. *Ref:* [Eji83] Sections 3. See the historical references [Che70, Sim68].
7. Riemann-Roch formula application to the minimal surfaces.
   Ref: [Eji83] Section 4.
8. Holomorphic curves in the 6-dimensional sphere. Hopf Problem. SU(3) structure on the 6-sphere (i.e. almost Hermitian).
   Ref: [Mad22] Section 1 and 2. See also [ABG+18].
9. Holomorphic curvature and torsion. Holomorphic Frenet equations. Holomorphic interpretation of the Torsion.
   Ref: [Mad22] Section 3. See also the original reference [Bry82].
10. Plateau Problem. *Ref:* [Sim68].
11. Bernstein Problem. *Ref:* [Sim68].
12. Minimal submanifolds of the Page metric. *Ref:* [KS21]

If time permits, the following additional topics may be chosen to present voluntarily.

- Weierstrass representation. *Ref:* [CM11]
- Lawson's minimal surface construction in the 3-sphere. *Ref:* [Law70]
- Douglas Problem. *Ref:* [DHS10] Chapter 8.
- Clifford torus and the Lawson conjecture with overview of its proof.
  Ref: [Bre13a, Bre13b]
- Selected topics from [Law80], [MP12], or [Mad22].

References

[ABG+18] Ilka Agricola, Giovanni Bazzoni, Oliver Goertsches, Panagiotis Konstantis, and Sönke Rollenske. On the history of the Hopf problem. *Differential Geom. Appl.*, 57:1–9, 2018.

[Bre13a] Simon Brendle. Embedded minimal tori in S^3 and the Lawson conjecture. *Acta Math.*, 211(2):177–190, 2013.

[Bre13b] Simon Brendle. Minimal surfaces in S^3: a survey of recent results. *Bull. Math. Sci.*, 3(1):133–171, 2013.

[Bry82] Robert L. Bryant. Submanifolds and special structures on the octonians. *J. Differential Geometry*, 17(2):185–232, 1982.

[Che70] Shiing Shen Chern. On the minimal immersions of the two-sphere in a space of constant curvature. In *Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969)*, pages 27–40. 1970.

[CM11] Tobias Holck Colding and William P. Minicozzi, II. *A course in minimal surfaces*, volume 121 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[DHS10] Ulrich Dierkes, Stefan Hildebrandt, and Friedrich Sauvigny. *Minimal surfaces*, volume 339 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, second edition, 2010. With assistance and contributions by A. Küster and R. Jakob.

[Eji83] Norio Ejiri. The index of minimal immersions of S^2 into S^{2n}. *Math. Z.*, 184(1):127–132, 1983.

[KS21] Mustafa Kalafat and Ramazan Sarı. On special submanifolds of the Page space. *Differential Geom. Appl.*, 74:101708, 2021.

[Law70] H. Blaine Lawson, Jr. Complete minimal surfaces in S^3. *Ann. of Math. (2)*, 92:335–374, 1970.

[Law80] H. Blaine Lawson, Jr. *Lectures on minimal submanifolds. Vol. I*, volume 9 of *Mathematics Lecture Series*. Publish or Perish, Inc., Wilmington, Del., second edition, 1980.

[Li12] Peter Li. *Geometric analysis*, volume 134 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.

[Mad22] Jesse Madnick. The Second Variation for Null-Torsion Holomorphic Curves in the 6-Sphere. *J. Geom. Anal.*, 32(12):Paper No. 295, 2022.

[MP12] William H. Meeks, III and Joaquín Pérez. *A survey on classical minimal surface theory*, volume 60 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2012.

[Sim68] James Simons. Minimal varieties in riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.