Graduate Seminar on Global Analysis (S4B3)
Summer Semester 2025
Dr. Mustafa Kalafat

Riemannian manifolds of special holonomy and calibrated geometries

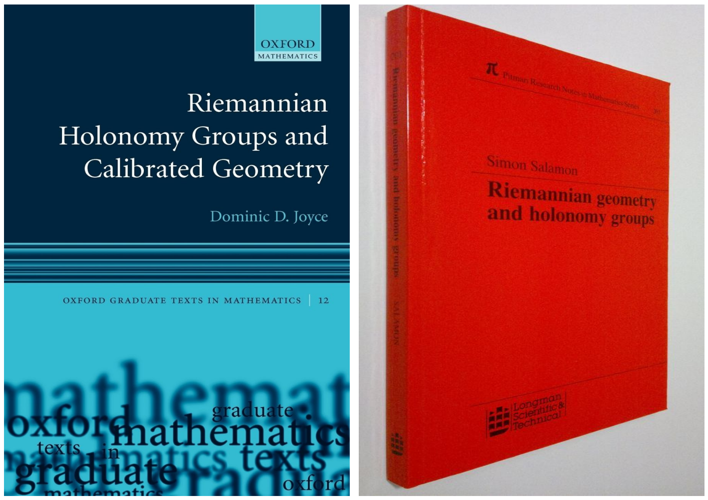

Figure 1: Two foundational texts.

In differential geometry, a calibrated manifold is a Riemannian manifold (M, g) of dimension n equipped with a differential p-form ϕ for some $0 \leq p \leq n$ which is a calibration, meaning that: ϕ is closed at each point, and it is a small multiple of the volume p-form at each p-subspace of the tangent bundle of M. That is to say, for any oriented p-dimensional subspace V of T_xM,

$$\phi|_V = \lambda \, vol_V \;\text{ for }\; \lambda \leq 1.$$

We have the set of maximal planes

$$G_x(\phi) = \{V < T_xM : \phi|_V = vol_V\}.$$

Let $G(\phi)$ be their union for x in M. A *calibrated submanifold* $N \subset M$ is the one which has tangent planes as maximal elements everywhere. That is abbreviated as $TN \subset \; G(\phi)$, actually $T_xN \in G(\phi)$ for each $x \in N$. Note that $G(\phi) \subset G_pTM$.

Calibrated Geometries are introduced by Harvey and Lawson in their foundational paper [HL82]. Calibrated submanifolds minimize volume within their homology class. So that they are a natural setting for a generalization of minimal submanifolds.

The *holonomy group* $Hol(g)$ of a Riemannian manifold determines the geometrical structures on M compatible with g. Thus, Berger's classification of Riemannian holonomy groups gives a list of interesting geometrical structures compatible with a Riemannian metric, and the aim of the subject is to study each such structure in depth. Most of the holonomy groups on Berger's list turn out to be important in string theory in theoretical physics. Given some class of mathematical objects, there is often a natural class of subobjects living inside them, such as groups and subgroups for instance. The natural subobjects of Riemannian manifolds with special holonomy are calibrated submanifolds - lower-dimensional, volume-minimizing submanifolds N in M compatible with the geometric structures coming from the holonomy reduction. So calibrated geometry is an obvious companion subject for Riemannian holonomy groups. Calibrated submanifolds are also important in string theory, as 'supersymmetric cycles' or 'branes'.

Among the manifolds with calibrated geometries are Kähler manifolds, Calabi-Yau manifolds, Quaternionic manifolds, Quaternion-Kähler manifolds, hyperkähler manifolds, G_2 and $Spin(7)$ manifolds.

The aim of this seminar is to study these spaces which are Einstein manifolds in most of the cases. We would like to take great care to motivate everything by working through numerous examples and by providing informal proof sketches as well as geometric interpretations and applications wherever possible.

Students are required to submit a manuscript that displays important aspects of their talks.

If you decide to take this seminar class, send me an e-mail with a preference of at least 3 topics from the syllabus below. Also include your full name, Matricul.no. and Uni-ID in the message.

Prerequisites: Advanced geometry 1.
Organizational(Prelm.) Meeting: February 19 2025, 14:00, in seminar room 0.011.
Seminar Time and Place: Wednesdays at 2:15-4 in seminar room 1.008.

Talks:

1. Manifolds and structure groups. Parallel transport. Unitary holonomy group. *Ref:* [Sal89] Chapters 1, 2 and 3. Alternatively, see the related chapters from [Joy07].

2. Riemannian and Kähler curvature.
   Ref: [Sal89] Chapter 4. Alternatively, see the related chapters from [Joy07].

3. Lie algebras and symmetric spaces.
   Ref: [Sal89] Chapter 5. See also [Kna02, FH91]

4. Representation theory. Weights and Roots. Tensor products.
   Ref: [Sal89] Chapter 6. See also [Kna02, FH91]

5. 4 dimensions. Self-dual Einstein manifolds. Twistor spaces.
   Ref: [Sal89] Chapter 7. Alternatively, see the related chapters from [Joy07].

6. Special Kähler manifolds: Calabi-Yau metrics, Hyperkähler metrics.
   Ref: [Sal89] Chapter 8. Alternatively, see the related chapters from [Joy07].

7. Quaternionic manifolds and classification theorems.
   Ref: [Sal89] Chapters 9 and 10. Alternatively, see the related chapters from [Joy07].

8. G_2 as a holonomy group.
   Ref: [Sal89] Chapter 11. Alternatively, see the related chapters from [Joy07].

9. $Spin(7)$ as a holonomy group. Clifford algebras.
   Ref: [Sal89] Chapter 12. Alternatively, see the related chapters from [Joy07].

10. Representation theory of complex Lie algebras $\mathfrak{sl}(n, \mathbb{C})$.
    Ref: [FH91] Chapters 11, 12 and 13.

11. Representation theory of the Lie algebra $\mathfrak{g}_2$.
    Ref: [FH91] Chapters 14 and 22.

12. Remarks on G_2 structures.
    Ref: [Bry06] First 5 sections. Alternatively, see the related chapters from [Joy07].

References

[Bry06] Robert L. Bryant. Some remarks on G_2-structures. In *Proceedings of Gökova Geometry-Topology Conference 2005*, pages 75–109. Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[HL82] Reese Harvey and H. Blaine Lawson, Jr. Calibrated geometries. *Acta Math.*, 148:47–157, 1982.

[Joy07] Dominic D. Joyce. *Riemannian holonomy groups and calibrated geometry*, volume 12 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2007.

[Kna02] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[Sal89] Simon Salamon. *Riemannian geometry and holonomy groups*, volume 201 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.