Proceedings of $13^{th}$ Gökova
Geometry-Topology Conference
pp. 106 – 122

# Deformations of Scalar-Flat Anti-Self-Dual Metrics and Quotients of Enriques Surfaces

*Mustafa Kalafat*

ABSTRACT. In this article, we prove that a quotient of a $K3$ surface by a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action does not admit any metric of positive scalar curvature. This shows that the scalar flat anti self-dual metrics (SF-ASD) on this manifold can not be obtained from a family of metrics for which the scalar curvature changes sign, contrary to the previously known constructions of this kind of metrics on manifolds of $b^+ = 0$.

## 1. Introduction

One of the most interesting features of the space of anti-self-dual or self-dual (ASD/SD) metrics on a manifold is that the scalar curvature can change sign on a connected component. That means, one can possibly join two ASD metrics of scalar curvatures of opposite signs by a 1-parameter family of ASD metrics. However, this is not the case, for example for the space of Einstein metrics. There, each connected component has a fixed sign for the scalar curvature.

As a consequence, contrary to the Einstein case, most of the examples of SF-ASD metrics are constructed by first constructing a family of ASD metrics. Then showing that there are metrics of positive and negative scalar curvature in the family, and guaranteeing that there is a scalar-flat member in this family. In the $b^+ = 0$ case actually this is the only way known to construct such metrics on a 4-manifolds. This paper presents an example of a SF-ASD Riemannian 4-manifold which is impossible to obtain by this kind of techniques since it does not have a positive scalar curvature deformation.

§2 introduces the ASD manifolds and the optimal metric problem, §3 reviews the known examples of SF-ASD metrics constructed by a deformation changing the sign of the scalar curvature, §4 introduces an action on the $K3$ surface and furnish the quotient manifold with a SF-ASD metric, §5 shows that the smooth manifold defined in §4 does not admit any positive scalar curvature (PSC) or PSC-ASD metric, finally §6 includes some related examples and remarks.

*Key words and phrases.* Self-Dual Metrics, Spin Structures, Dirac Operator, Kähler Manifolds, Algebraic Surfaces.

## 2. Weyl Curvature Tensor and Optimal Metrics

Let $(M, g)$ be an oriented Riemannian n-manifold. Then by raising the indices, the Riemann curvature tensor at any point can be viewed as an operator $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ hence an element of $S^2\Lambda^2 M$. It satisfies the algebraic Bianchi identity hence lies in the vector space of *algebraic curvature tensors*. This space is an $O_n$-module and has an orthogonal decomposition into irreducible subspaces for $n \geq 4$. Accordingly the Riemann curvature operator decomposes as:

$$\mathcal{R} = U \oplus Z \oplus W$$

where

$$U = \frac{s}{2n(n-1)} g \bullet g \quad \text{and} \quad Z = \frac{1}{n-2} \overset{\circ}{Ric} \bullet g$$

$s$ is the scalar curvature, $\overset{\circ}{Ric} = Ric - \frac{s}{n} g$ is the trace-free Ricci tensor, "$\bullet$" is the Kulkarni-Nomizu product, and $W$ is the *Weyl Tensor* which is defined to be what is left over from the first two piece.

Let $(M, g)$ be an oriented Riemannian manifold of dimension $n$. We have a linear transformation between the bundles of exterior forms called the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$. It is the unique vector bundle isomorphism between $\binom{n}{p}$-dimensional vector bundles defined by

$$\alpha \wedge (*\beta) = g(\alpha, \beta) dV_g$$

where $\alpha, \beta \in \Lambda^p, dV_g$ is the canonical n-form of $g$ satisfying $dV_g(e_1, , e_n) = 1$ for any oriented orthonormal basis $e_1, , e_n$. $*$ is defined pointwise but it takes smooth forms to smooth forms, so induces a linear operator $* : \Gamma(\Lambda^p) \to \Gamma(\Lambda^{n-p})$ between infinite dimensional spaces. Notice that $*1 = dV_g, *dV_g = 1$ and $*^2 = (-1)^{p(n-p)} Id_{\wedge^p}$. [Bes, AHS, War]

If $n$ is even, star operates on the middle dimension with $*^2 = (-1)^{n/2} Id_{\wedge^{n/2}}$. Moreover it is conformally invariant in dimension $n/2$: If we rescale the metric by a scalar $\lambda$, $\widetilde{g} = \lambda g$, $dV_g = \lambda^{n/2} dV_g$ so that their product remains unchanged on $n/2$-forms since the inner product on the cotangent vectors multiplied by $\lambda^{-1}$.

If $n = 2$, $*$ acts on $\Lambda^1$ or $TM^*$ as well as $TM$ by duality with $*^2 = -Id_{TM}$. So it defines a complex structure on a surface.

The case we are interested is $n = 4$, i.e. we have a Riemannian 4-Manifold and $* : \Lambda^2 \to \Lambda^2$ with $*^2 = Id_{\wedge^2}$ and we have eigenspaces $E_x(\pm 1)$ over each point $x$ denoted $(\Lambda^2_{\pm})_x$ and the bundle $\Lambda^2$ splits as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. We call these bundles, bundle of *self dual , anti self dual two forms* respectively. The splitting of two forms turns out to have a great influence on the geometry of the 4-manifold because of the fact that the Riemann curvature tensor can be considered as an operator on two forms and so it also has a corresponding splitting. The decomposition of the space of two forms yields a decomposition of any operator acting on this space. In particular $W_{\pm} : \Lambda^2_{\pm} \to \Lambda^2_{\pm}$ called self-dual and anti-self-dual pieces of the Weyl curvature operator. And we call $g$ to be *self-dual* (or *anti-self-dual*) *metric* if $W_-$(or $W_+$) vanishes. If one reverses the orientation,

$W_-$ and $W_+$ are interchanged, so a SD manifold can be considered as an ASD manifold with the reverse orientation.

4-manifolds which support self-dual metrics are abundant. In particular [Taubes] shows that for any smooth 4-manifold, the blow up $M\#k\overline{\mathbb{CP}}_2$ admits an ASD metric for sufficiently large $k$. Besides, there is even a connected sum theorem : under reasonable circumstances, Donaldson and Friedman proves in their seminal paper [DonFried] that the connected sums of SD manifolds are also SD. It is a pity that these deep and beautiful results of Taubes, Donaldson and Friedman are not as well known as their other results among the general geometry-topology community. Even though the result of Donaldson and Friedman is very strong, it has a small drawback. It does not tell anything about the curvature of the connected sum metric. To remedy this situation, the author proved that one can generalize their construction to the positive scalar curvature case, i.e. under the same conditions of [DonFried] the connected sums of 4-manifolds that admit PSC-SD metrics again admit PSC-SD metrics. See [Kalafat] for details.

Finally, some motivation on why to study SD/ASD metrics is in order. SF-ASD metrics are solutions to the *optimal metric* problem. Optimal metric problem is a struggle to find a "best" metric for a smooth manifold. Historically, the geometers are interested in constant sectional curvature spaces. As soon as these spaces are classified, there is a question of what to do with the manifolds that do not admit any constant sectional curvature. Some of them are metrized by Einstein metrics, which have constant Ricci curvature. However there are still manifolds which do not admit any Einstein metrics. At this point SF-ASD metrics come into the picture. More precisely :

**Definition 2.1** ([LeOM])**.** A Riemannian metric on a smooth 4-manifold $M$ is called an *optimal metric* if it is the absolute minimum of the $L^2$ norm of the Riemann Curvature tensor on the space of metrics

$$\mathcal{K}(g) = \int_M |\mathcal{R}_g|^2 dV_g.$$

Using the orthogonal decomposition it is equal to

$$\mathcal{K}(g) = \int_M \frac{s^2}{24} + \frac{|\overset{\circ}{Ric}|^2}{2} + |W|^2 \, dV_g \ .$$

On the other hand, the generalized Gauss-Bonnet Theorem and the Hirzebruch Signature Theorem express the Euler characteristic $\chi$ and signature $\tau$ respectively as

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{s^2}{24} + |W|^2 - \frac{|\overset{\circ}{Ric}|^2}{2} \, dV_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \; dV_g \; .$$

Combining the two in $\mathcal{K}$ gives the following expression of $\mathcal{K}$

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \frac{s^2}{24} + 2|W_+|^2 \; dV_g \; .$$

This yields

**Proposition 2.1** ([LeOM])**.** *Let $M$ be a smooth compact oriented 4-manifold. If $M$ admits a SF-ASD metric then this metric is optimal. In this case all other optimal metrics are SF-ASD, too.*

For further information on the optimal metric problem, we suggest the excellent survey article [LeOM] by C. LeBrun.

## 3. Constructions of SF-ASD Metrics

Here we review some of the constructions for SF-ASD metrics on 4-manifolds. We begin with

**Theorem 3.1** (LeBrun[LeOM])**.** *For all integers $k \geq 6$, the manifold*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k-many}$$

*admits a 1-parameter family of real analytic ASD conformal metrics $[g_t]$ for $t \in [0,1]$ such that $[g_0]$ contains a metric of $s > 0$ on the other hand $[g_1]$ contains a metric of $s < 0$.*

Now we are going to state the *strong maximum principle of Hopf's*. Before, we have a definition. Consider the differential operator $L_c = \sum_{i,j=1}^{n} a^{ij}(x_1..x_n)\frac{\partial^2}{\partial x_i \partial x_j}$ arranged so that $a^{ij} = a^{ji}$. It is called *elliptic* ([PrWe]*p56* ) at a point $x = (x_1..x_n)$ if there is a positive quantity $\mu(x)$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \mu(x)\sum_{i=1}^{n} {\xi_i}^2$$

for all $n$-tuples of real numbers $(\xi_1..\xi_n)$. The operator is said to be uniformly elliptic in a domain $\Omega$ if the inequality holds for each point of $\Omega$ and if there is a positive constant $\mu_0$ such that $\mu(x) \geq \mu_0$ for all $x$ in $\Omega$. Ellipticity of a more general second order operator is defined via its second order term. In the matrix language, the ellipticity condition asserts that the symmetric matrix $[a^{ij}]$ is positive definite at each point $x$.

**Lemma 3.2** (Hopf's strong maximum principle , [PrWe]p64)**.** *Let $u$ satisfy the differential inequality*

$$(L_c + h)u \geq 0 \textit{ with } h \leq 0$$

*where $L_c$ is uniformly elliptic in $\Omega$ and coefficients of $L_c$ and $h$ bounded. If $u$ attains a nonnegative maximum at an interior point of $\Omega$, then $u$ is constant.*

**Corollary 3.3** (LeBrun[LeOM]). *For all integers $k \geq 6$, the connected sum $k\overline{\mathbb{CP}}_2$ admits scalar-flat anti-self-dual (SF-ASD) metrics.*[1]

*Proof.* Let $h_t \in [g_t]$ be a smooth family of metrics representing the smooth family of conformal classes $[g_t]$ constructed in [LeOM]. We know that the smallest eigenvalue $\lambda_t$ of the Yamabe Laplacian $(\Delta + s/6)$ of the metric $h_t$ exists, and is a continuous function of $t$. It measures the sign of the conformally equivalent constant scalar curvature metric [LP].

But the theorem(3.1) tells us that $\lambda_0$ and $\lambda_1$ has opposite signs. Then there is some $c \in [0,1]$ for which $\lambda_c = 0$. Let $u$ be the eigenfunction corresponding to the eigenvalue 0, for the Yamabe Laplacian of $h_c$ , i.e. $(\Delta + s/6)u = 0$. Rescale it by a constant so that it has unit integral.

Rescale the metric $h_c$ so that it has constant scalar curvature [LP]. We have three cases for the scalar curvature, positive, zero or negative. If it is zero then we are done. Suppose $s_c = s > 0$. Since $u$ is a continuous function on the compact manifold, it has a minimum, say at $m$. Choose the normal coordinates around there, so that $\Delta u(m) = -\sum_{k=1}^4 \partial^2 u(m)$. Second partial derivatives are greater than or equal to zero, $\Delta u(m) \leq 0$ so $u(m) = -\frac{6}{s}\Delta u(m) \geq 0$. Assume $u(m) = 0$. Then the maximum of $-u$ is attained and it is nonnegative with $(-\Delta - s/6)(-u) = 0 \geq 0$. So the strong maximum principle (3.2) is applicable and $-u \equiv 0$, which is not an eigenfunction. So $u$ is a positive function. For a conformally equivalent metric $\tilde{g}$, the new scalar curvature $\tilde{s}$ is computed to be [Bes]

$$\tilde{s} = 6u^{-3}(\Delta + s/6)u$$

in terms of $s$. Thus $\widetilde{g} = u^2 h_c$ is a scalar-flat anti-self-dual metric on $k\overline{\mathbb{CP}}_2$ for any $k \geq 6$. The negative scalar curvature case is treated similarly. □

Another construction tells us

**Theorem 3.4** ([Kim]). *There exist a continuous family of self-dual metrics on a connected component of the moduli space of self-dual metrics on*

$$l(S^3 \times S^1)\# m\mathbb{CP}_2 \text{ for any } m \geq 1 \text{ and for some } l \geq 2$$

*which changes the sign of the scalar curvature.*

## 4. SF-ASD Metric on the Quotient of Enriques Surface

In this section we are going to describe what we mean by $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the scalar-flat anti-self-dual (SF-ASD) metric on it.

[1] Quite recently, LeBrun and Maskit announced that they have extended this result to the case $k = 5$ with similar techniques, which is the minimal number for these type of connected sums according to [LeSD].

Let $A$ and $B$ be real $3 \times 3$ matrices. For $x, y \in \mathbb{C}^3$, consider the algebraic variety $V_{2,2,2} \subset \mathbb{CP}_5$ given by the equations

$$\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0 \ , \ i = 1, 2, 3$$

or more precisely,

$$A_1^1 x_1^2 + A_1^2 x_2^2 + A_1^3 x_3^2 + B_1^1 y_1^2 + B_1^2 y_2^2 + B_1^3 y_3^2 = 0$$
$$A_2^1 x_1^2 + A_2^2 x_2^2 + A_2^3 x_3^2 + B_2^1 y_1^2 + B_2^2 y_2^2 + B_2^3 y_3^2 = 0$$
$$A_3^1 x_1^2 + A_3^2 x_2^2 + A_3^3 x_3^2 + B_3^1 y_1^2 + B_3^2 y_2^2 + B_3^3 y_3^2 = 0$$

For generic $A$ and $B$, this is a complete intersection of three nonsingular quadric hypersurfaces. By the Lefschetz hyperplane theorem, it is simply connected, and

$$K_{V_2} = K_{\mathbb{P}^5} \otimes [V_2^{\mathbb{P}^5}] = \mathcal{O}(-6) \otimes \mathcal{O}(1)^{\otimes 2} = \mathcal{O}(-4)$$

since $[V_2]_h = 2[H]_h$ and taking Poincare duals, similarly

$$K_{V_{2,2}} = K_{V_2} \otimes [V_{2,2}^{\mathbb{P}^5}] = \mathcal{O}(-4) \otimes \mathcal{O}(2) = \mathcal{O}(-2)$$

$$K_{V_{2,2,2}} = K_{V_{2,2}} \otimes [V_{2,2,2}^{\mathbb{P}^5}] = \mathcal{O}(-2) \otimes \mathcal{O}(2) = \mathcal{O}$$

finally. So the canonical bundle is trivial. $V$ is a $K3$ Surface. We define the commuting involutions $\sigma^{\pm}$ by

$$\sigma^+(x, y) = (x, -y) \text{ and } \sigma^-(x, y) = (\bar{x}, \bar{y})$$

and since we arranged $A$ and $B$ to be real, $\sigma^{\pm}$ both act on $V$.

At a fixed point of $\sigma^+$ on $V$, we have $y_j = -y_j = 0$, so $\sum_j A_i^j x_j^2 = 0$. So if we take an invertible matrix $A$, these conditions are only satisfied for $x_j = y_j = 0$ which does not correspond to a point, so $\sigma^+$ is free and holomorphic. At a fixed point of $\sigma^-$ on $V$, $x_j$'s and $y_j$'s are all real. If $A_1^j, B_1^j > 0$ for all $j$ then $\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0$ forces $x_j = y_j = 0$ making $\sigma^-$ free. At a fixed point $\sigma^- \sigma^+$ on $V$, $x_j = \bar{x}_j$ and $y_j = -\bar{y}_j$, so $x_j$'s are real and $y_j$'s are purely imaginary. Then $y_j^2$ is a negative real number. So if we choose $A_2^j > 0$ and $B_2^j < 0$, this forces $x_j = y_j = 0$, again we obtain a free action for $\sigma^- \sigma^+$. Thus choosing $A$ and $B$ within these circumstances $\sigma^{\pm}$ generate a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action and we define $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be the quotient of $K3$ by this free action. We have

$$\chi = \sum_{k=0}^{4} (-1)^k b_k = 2 - 2b_1 + b_2 = 2 + (2b^+ - \tau) \text{ hence } b^+ = (\chi + \tau - 2)/2$$

so, $b^+(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = (24/4 - 16/4 - 2)/2 = 0$, a special feature of this manifold.

Next we are going to furnish this quotient manifold with a Riemannian metric. For that purpose, there is a crucial observation [HitEin] that, for any free involution on $K3$, there exists a complex structure on $K3$ making this involution holomorphic, so the quotient is a complex manifold. We begin by stating the

**Theorem 4.1** (Calabi-Yau[Ca, Yau, GHJ, Joyce]). *Let $(M, \omega)$ be a compact Kähler $n$-manifold. Let $\rho$ be a $(1,1)$-form belonging to the class $2\pi c_1(M)$ so that it is closed. Then, there exists a unique Kähler metric with form $\omega'$ which is in the same class as in $\omega$, whose Ricci form is $\rho$.*

Intuitively, one can slide the Kähler form $\omega$ in its cohomology class and obtain any desired reasonable Ricci form $\rho$.

*Remark* 4.2. Since $c_1(K3) = 0$ in our case, taking $\rho \equiv 0$ gives us a Ricci-Flat(RF) metric on the $(K3, \omega)$ surface, the *Calabi-Yau metric.* This metric is hyperkählerian because of the following reason: The holonomy group of Kähler manifolds are a subgroup of $U_2$. However, Ricci-flatness reduces the holonomy since harmonic forms are parallel because of the Weitzenböck Formula for the Hodge/modern Laplacian on 2-forms (5.6). Scalar flatness and non-triviality of $b^+$ is to be checked. $b_1(K3) = 0$ implies $b^+(K3) = (24 - 16 - 2)/2 = 3$, which is nonzero. Actually $b^+$ is nontrivial for any Kähler surface since the Kähler form is harmonic and self-dual. Harmonic parallel forms are kept fixed by the holonomy group, a fact that imposes a reduction from $U_2$ to $SU_2$ which is the next possible option and isomorphic to $Sp_1$ in this dimension, hence the Calabi-Yau metric is hyperkähler. Alternatively one can see that the holomorphic forms are also parallel by a similar argument, another reason to reduce the holonomy.

So we have at least three almost complex structures $I, J, K$, parallel with respect to the Riemannian connection. By duality we regard these as three linearly independent self-dual 2-forms, parallelizing $\Lambda_2^+$. So any parallel $\Lambda_2^+$ form on $K3$ defines a complex structure after normalizing. In other words $aI + bJ + cK$ defines a complex structure for the constants satisfying $a^2 + b^2 + c^2 = 1$, i.e the normalized linear combination. On the other hand

$$b_1(K3/\mathbb{Z}_2) = b_1(K3) = 0\ ,\ b^+(K3/\mathbb{Z}_2) = (12 - 8 - 2)/2 = 1.$$

Since the pullback of harmonic forms stay harmonic, the generating harmonic 2-form on $K3/\mathbb{Z}_2$ comes from the universal cover, so is fixed by the $\mathbb{Z}_2$ action. It is also a parallel self-dual form so its normalization is then a complex structure left fixed by $\mathbb{Z}_2$. So the quotient is a complex surface with $b_1 = 0$ and $2c_1 = 0$ implying that it is an *Enriques Surface.*

So we saw that any involution or $\mathbb{Z}_2$-action can be made holomorphic by choosing the appropriate complex structure on $K3$. In particular by changing the complex structure, $\sigma^-$ becomes holomorphic, too and then both $K3/\mathbb{Z}_2^{\pm}$ are complex manifolds, i.e. Enriques Surfaces, for $\mathbb{Z}_2^{\pm} = \langle \sigma^{\pm} \rangle$.

*Remark* 4.3. Even though we managed to make $\sigma^+$ and $\sigma^-$ into holomorphic actions by modifying the complex structure, it is impossible to provide a complex structure according to which they are holomorphic at the same time. The reason is that, in such a situation the quotient $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ would be a complex manifold. On the other hand the *Noether's*

*Formula* [Beauville]

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi(S)) = \frac{1}{12}(c_1^2 + c_2)[S]$$

holds for any compact complex surface [BPV] as a consequence of the Hirzebruch-Riemann-Roch Theorem. It produces a non-integer holomorphic Euler characteristic $\frac{1}{12}\frac{24}{4} = \frac{1}{2}$.

Now consider another metric on $K3$ : the restriction of the *Fubini-Study metric* on $\mathbb{CP}^5$ obtained from the Kähler form

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log|(x_1, x_2, x_3, y_1, y_2, y_3)|^2$$

We also denote the restriction metric by $g_{FS}$. It is clear that $\sigma^{\pm}$ leave this form invariant, hence they are isometries of $g_{FS}$. This is not the metric we are seeking for. This metric has all sectional curvatures lying in the interval $[1, 4]$ and is actually Einstein, i.e. $Ric = 6g$ with constant positive scalar curvature equal to 2 [Pet]p84. Let $g_{RF}$ be the Ricci-Flat Yau metric (4.1) taking $\rho \equiv 0$ with Kähler form cohomologous to $\omega_{FS}$. We will show that this metric is invariant under $\sigma^{\pm}$ and projects down to a metric on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Scalar flatness and being ASD are equivalent notions for Kähler metrics [LeSD], and the local structure does not change under isometric quotients which makes the quotient SF-ASD.

Since $\sigma^+$ is holomorphic, the pullback form $\sigma^{+*}\omega_{RF}$ is Kähler and the equalities

$$[\sigma^{+*}\omega_{RF}] = \sigma^{+*}[\omega_{RF}] = \sigma^{+*}[\omega_{FS}] = [\sigma^{+*}\omega_{FS}] = [\omega_{FS}]$$

show that it is cohomologous to the Fubini-Study form. Ricci curvature is preserved and is zero, hence by Calabi uniqueness (4.1) we get $\sigma^{+*}g_{RF} = g_{RF}$. Dealing with the anti-holomorphic involution needs a little more care. Think $\sigma^- : K3 \to K3$ as a diffeomorphism. The pullback of a Kähler metric is Kähler with respect to the pullback complex structure. The anti-holomorphicity relation relates the two complex structures by $\sigma^-_* J_1 = -J_2\sigma^-_*$. The pullback Kähler form $\widetilde{\omega}_n = \omega_{\sigma^{-*}g_{FS}} = -\sigma^{-*}\omega_{RF}$ since

$$\begin{aligned}\widetilde{\omega}_n(u, v) = \sigma^{-*}g_{RF}(J_1 u, v) &= g_{RF}(\sigma^-_* J_1 u, \sigma^-_* v) = g_{RF}(-J_2\sigma^-_* u, \sigma^-_* v)\\ &= -\omega_{RF}(\sigma^-_* u, \sigma^-_* v) = -\sigma^{-*}\omega_{RF}(u, v),\end{aligned}$$

and hence,

$$[\widetilde{\omega}_n] = [-\sigma^{-*}\omega_{RF}] = -\sigma^{-*}[\omega_{RF}] = -\sigma^{-*}[\omega_{FS}] = -[\sigma^{-*}\omega_{FS}] = -[\omega_{FS}].$$

But this is the form of $\sigma^{-*}g_{FS}$ with respect to the pullback complex structure which is the conjugate(negative) of the original one. Looking from the real point of view, once we have a Kähler metric $g$, it has a Kähler form corresponding to each supported complex structure on the manifold. Once the complex structure is chosen, the form is obtained by lowering an index

$$\omega_{ab} = \omega(\partial_a, \partial_b) = g(J\partial_a, \partial_b) = g(J_a{}^c\partial_c, \partial_b) = J_a{}^c\, g(\partial_c, \partial_b) = J_a{}^c\, g_{cb} = J_{ab}.$$

So, the form and the complex structure are equivalent from the tensorial point of view. If we conjugate(negate) the complex structure, we should replace the form with its negative. Returning to our case, $\widetilde{\omega}_n$ is the form corresponding to the pullback, hence to the conjugate complex structure. We take its negative to obtain the one corresponding to the original complex structure. So the corresponding form is going to be $\widetilde{\omega} = - - \omega_{FS}$ which is $\omega_{FS}$ , and again the Calabi uniqueness (4.1) implies $\sigma^{-*} g_{RF} = g_{RF}$.

*Remark* 4.4. There is an alternative argument in [McI]p894 which appears to have a gap: "Fubini-Study metric projects down to the metrics $g^{\pm}_{FS}$ on $K3/\mathbb{Z}_2^{\pm}$. Let $h^{\pm}$ be the Calabi-Yau metric(4.1) on $K3/\mathbb{Z}_2^{\pm}$ with Kähler form cohomologous to that of $g^{\pm}_{FS}$. To remedy the ambiguity in the negative side, keep in mind that, $\sigma^-$ fixes the metric and the form on $K3$, though the quotient is not a Kähler manifold initially since it is not a complex manifold, it is locally Kähler. We arrange the complex structure of $K3$ to provide a complex structure to the form, so the quotient manifold is Kähler. Now we have two Kähler metrics on the quotient (for different complex structures) but we do not know much about their curvatures, and want to make it Ricci-Flat, so we use the Calabi-Yau argument. Since $c_1(K3/\mathbb{Z}_2^{\pm}) = 0$ with real coefficients, we pass to the Calabi-Yau metric for $\rho \equiv 0$. $\pi^{\pm}$ denoting the quotient maps, the pullback metrics $\pi^{\pm *} h^{\pm}$ are both Ricci-Flat-Kähler(RFK) metrics on $K3$ with Kähler forms cohomologous to that of $g_{FS}$. Their Ricci forms are both zero. By the uniqueness(4.1) of the Yau metric we have $\pi^{+*} h^+ = \pi^{-*} h^-$. Hence this is a Ricci-Flat Kähler metric on $K3$ on which both $\sigma^{\pm}$ act isometrically. This metric therefore projects down to a Ricci-Flat metric on our manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$." The problem is that the pullback metrics $\pi^{\pm *} h^{\pm}$ are Kähler metrics with cohomologous Kähler forms, however they are Kählerian with respect to different complex structures. So the Calabi uniqueness (4.1) can not be applied directly.[2]

## 5. Weitzenböck Formulas

Now we are going to show that the smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any positive scalar curvature metric. For that purpose we state the Weitzenböck Formula for the Dirac Operator on spin manifolds. Before that we introduce some notation together with some ingredients of the formula.

For any vector bundle $E$ over a Riemannian Manifold $M$, the Levi-Civita connection is going to be the linear map we denote by $\nabla : \Gamma(E) \to \Gamma(Hom(TM, E))$. Then we get the adjoint $\nabla^* : \Gamma(Hom(TM, E)) \to \Gamma(E)$ defined implicitly by

$$\int_M \langle \nabla^* S, s \rangle dV = \int_M \langle S, \nabla s \rangle dV$$

and we define the *connection Laplacian* of a section $s \in \Gamma(E)$ by their composition $\nabla^* \nabla s$. Notice that the harmonic sections are parallel for this operator. Using the metric, we can express its action as :

**Proposition 5.1** ([Pet]p179). *Let $(M, g)$ be an oriented Riemannian manifold, $E \to M$ a vector bundle with an inner product and compatible connection. Then*

$$\nabla^* \nabla s = -tr \nabla^2 s$$

[2]Thanks to the second referee for pointing out this delicate issue.

*for all compactly supported sections of $E$.*

*Proof.* First we need to mention the second covariant derivatives and then the integral of the divergence. We set

$$\nabla^2 K(X,Y) = (\nabla\nabla K)(X,Y) = (\nabla_X \nabla K)(Y).$$

Then using the fact that $\nabla_X$ is a derivation commuting with every contraction: [KN1]p124

$$\begin{aligned}\nabla_X \nabla_Y K &= \nabla_X C(Y \otimes \nabla K) = C\nabla_X (Y \otimes \nabla K)\\ &= C(\nabla_X Y \otimes \nabla K + Y \otimes \nabla_X \nabla K)\\ &= \nabla_{\nabla_X Y} K + (\nabla_X \nabla K)(Y)\\ &= \nabla_{\nabla_X Y} K + \nabla^2 K(X,Y)\end{aligned}$$

hence $\nabla^2 K(X,Y) = \nabla_X \nabla_Y K - \nabla_{\nabla_X Y} K$ for any tensor $K$. That is how the second covariant derivative is defined . Higher covariant derivatives are defined inductively.

For the divergence, remember that

$$(divX)dV_g = \mathcal{L}_X dV_g,$$

which is taken as a definition sometimes[KN1]p281. After combining this with the Cartan's Formula: $\mathcal{L}_X dV = di_X dV + i_X d(dV) = di_X dV$; Stokes' Theorem yields that $\int_M (divX)dV = \int_M \mathcal{L}_X dV = \int_M d(i_X dV) = \int_{\partial M} i_X dV = 0$ for a compact manifold without boundary. This is actually valid even for a noncompact manifold together with a compactly supported vector field.

Now take an open set on $M$ with an orthonormal basis $\{E_i\}_{i=1}^n$. Let $s_1$ and $s_2$ be two sections of $E$ compactly supported on the open set. We reduce the left-hand side via multiplying by $s_2$ as follows:

$$\begin{aligned}(\nabla^*\nabla s_1, s_2)_{L^2} = \int_M \langle \nabla^*\nabla s_1, s_2\rangle dV = \int_M \langle \nabla s_1, \nabla s_2\rangle dV &= \int_M tr((\nabla s_1)^*\nabla s_2) dV\\ &= \sum_{i=1}^n \int_M \langle (\nabla s_1)^* \nabla s_2(E_i), E_i\rangle dV\\ &= \sum \int_M \langle (\nabla s_1)^* \nabla_{E_i} s_2, E_i\rangle dV\\ &= \sum \int_M \langle \nabla_{E_i} s_2, \nabla s_1(E_i)\rangle dV\\ &= \sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2\rangle dV.\end{aligned}$$

Define a vector field $X$ by $g(X,Y) = \langle \nabla_Y s_1, s_2\rangle$. Divergence of this vector field is

$$\begin{aligned}divX = -d^*(X^\flat) = tr\nabla X = \textstyle\sum_{i=1}^n \langle \nabla_{E_i} X, E_i\rangle &= \textstyle\sum (E_i\langle X, E_i\rangle - \langle X, \nabla_{E_i} E_i\rangle)\\ &= \textstyle\sum (E_i\langle \nabla_{E_i} s_1, s_2\rangle - \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2\rangle).\end{aligned}$$

We know that its integral is zero, so our expression continues to evolve as

$$\sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2 \rangle dV - \int_M (divX) dV$$

$$= \sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2 \rangle dV - \sum \int_M (E_i \langle \nabla_{E_i} s_1, s_2 \rangle - \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$

$$= \sum \int_M (-\langle \nabla_{E_i} \nabla_{E_i} s_1, s_2 \rangle + \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$

$$= \sum \int_M \langle -\nabla^2 s_1(E_i, E_i), s_2 \rangle dV$$

$$= -\int_M \langle \sum \langle \nabla^2 s_1(E_i), E_i \rangle, s_2 \rangle dV$$

$$= \int_M \langle -tr \nabla^2 s_1, s_2 \rangle dV$$

$$= (-tr \nabla^2 s_1, s_2)_{L^2}.$$

So we established that $\nabla^* \nabla s_1 = -tr \nabla^2 s_1$ for compactly supported sections in an open set.

□

**Theorem 5.2** (Atiyah-Singer Index Theorem[LM]p256,[MoSW]p47)**.** *Let $M$ be a compact spin manifold of dimension $n = 2m$. Then, the index of the Dirac operator is given by*

$$ind(\not{D}^+) = \hat{A}(M) = \hat{\boldsymbol{A}}(M)[M].$$

*More generally, if $E$ is any complex vector bundle over $M$, the index of $\not{D}_E^+ : \Gamma(\mathbb{S}_\pm \otimes E) \to \Gamma(\mathbb{S}_\mp \otimes E)$ is given by*

$$ind(\not{D}_E^+) = \{ch(E) \cdot \hat{\boldsymbol{A}}(M)\}[M].$$

*For $n = 4$, $\hat{\boldsymbol{A}}(M) = 1 - p_1/24$ and the first formula reduces to*

$$ind(\not{D}^+) = \hat{A}(M) = \int_M -\frac{p_1(M)}{24} = -\frac{\tau(M)}{8}$$

*by the Hirzebruch Signature Theorem.*

Let us explain the ingredients beginning with the cohomology class $\hat{\mathbf{A}}(M)$. Consider the power series of the following function[Fr]p108 :

$$\frac{t/2}{\sinh t/2} = \frac{t}{e^{t/2} - e^{-t/2}} = 1 + A_2 t^2 + A_4 t^4 + \dots$$

where we compute the coefficients as

$$A_2 = -\frac{1}{24} \ , \ A_4 = \frac{7}{10 \cdot 24 \cdot 24} = \frac{7}{5760}.$$

Consider the Pontrjagin classes $p_1 ... p_k$ of $M^{4k}$. Represent these as the elementary symmetric functions in the squares of the formal variables $x_1 \cdots x_k$:

$$x_1^2 + \cdots + x_k^2 = p_1 \ , \ \cdots \ , \ x_1^2 x_2^2 \cdots x_k^2 = p_k$$

Then $\prod_{i=1}^{k} \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$ is a symmetric power series in the variables $x_1^2 \cdots x_k^2$, hence defines a polynomial in the Pontrjagin classes. We call this polynomial $\hat{\mathbf{A}}(M)$

$$\hat{\mathbf{A}}(M) = \prod_{i=1}^{k} \frac{x_i/2}{\sinh x_i/2}.$$

In lower dimensions we have

$$\hat{\mathbf{A}}(M^4) = 1 - \frac{1}{24} p_1 \ , \ \hat{\mathbf{A}}(M^8) = 1 - \frac{1}{24} p_1 + \frac{7}{5760} p_1^2 - \frac{1}{1740} p_2.$$

If the manifold has dimension $n = 4k+2$, again it has $k$ Pontrjagin classes, and we define the polynomial $\hat{\mathbf{A}}(M^{4k+2})$ by the same formulas.

Secondly, we know that $\not{D}^+ : \Gamma(\mathbb{S}_+) \to \Gamma(\mathbb{S}_-)$ is an elliptic operator, so its kernel is finite dimensional and its image is a closed subspace of finite codimension. The *index* of an elliptic operator is defined to be $dim(kernel) - dim(cokernel)$. Actually in our case $\not{D}^+$ and $\not{D}^-$ are formal adjoints of each other: $(\not{D}^+\psi, \eta)_{L^2} = (\psi, \not{D}^-\eta)_{L^2}$ for $\psi, \eta$ compactly supported sections [LM]p114 , [MoSW]p42. Consequently the index becomes $dim(ker)\not{D}^+) - dim(ker\not{D}^-)$.

This index is computed from the symbol in the following way. Consider the pullback of $\mathbb{S}_\pm$ to the cotangent bundle $T^*M$. The symbol induces a bundle isomorphism between these bundles over the complement of the zero section of $T^*M$. In this way the symbol provides an element in the relative K-theory of $(T^*M, T^*M - M)$. The Atiyah-Singer Index Theorem computes the index from this element in the relative K-theory. In the case of the Dirac operator the index is $\hat{A}(M)$, the so-called *A-hat genus* of $M$.

Now we are ready to state our main tool

**Theorem 5.3** (Weitzenböck Formula[Pet]p183,[Bes]p55)**.** *On a spin Riemannian manifold, consider the Dirac operator $\not{D} : \Gamma(\mathbb{S}_\pm) \to \Gamma(\mathbb{S}_\mp)$. The Dirac Laplacian can be expressed in terms of the connection/rough Laplacian as*

$$\not{D}^2 = \nabla^*\nabla + \frac{s}{4}$$

*where $\nabla$ is the Riemannian connection.*

Finally we state and prove our main result :

**Theorem 5.4.** *The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any metric of positive scalar curvature(PSC).*

*Proof.* If $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ admits a metric of PSC then $K3$ is also going to admit such a metric because one pulls back the metric of the quotient, and obtain a locally identical metric on which the PSC survives.

So we are going to show that the $K3$ surface does not admit any metric of PSC. First of all the canonical bundle of $K3$ is trivial so that $c_1(K3) = 0 = w_2(K3)$ implying that it is a spin manifold.
By the Atiyah-Singer Index Theorem (5.2),

$$ind(\not{D}^+) = \hat{\mathbf{A}}(M)[M] = -\frac{\tau(M)}{8} = 2$$

for the $K3$ Surface. Since it is equal to $dim(ker) - dim(coker)$, this implies that the $dim(ker\not{D}^+) \geq 2$.

Let $\psi \in ker\not{D}^+ \subset \Gamma(\mathbb{S}_+)$ and consider its image $(\psi, 0)$ in $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$. Then $\not{D}^2(\psi, 0) = 0$ since $\not{D} = \not{D}^+ \oplus \not{D}^-$. Abusing the notation as $\psi = (\psi, 0)$ the spin Weitzenböck Formula (5.3) implies

$$0 = \nabla^*\nabla\psi + \frac{s}{4}\psi.$$

Taking the inner product with $\psi$ and integrating over the manifold yields

$$0 = (\nabla^*\nabla\psi, \psi)_{L^2} + (\frac{s}{4}\psi, \psi)_{L^2} = (\nabla\psi, \nabla\psi)_{L^2} + \frac{s}{4}(\psi, \psi)_{L^2} = \int_M (|\nabla\psi|^2 + \frac{s}{4}|\psi|^2) dV$$

and $s > 0$ implies that $|\nabla\psi| = |\psi| = 0$ everywhere, hence $\psi \equiv 0$. So $ker\not{D}^+ = 0$, which is not the case.
Notice that $s \geq 0$ and $s(p) > 0$ for some point is also enough for the conclusion because then $\psi$ would be parallel and zero at some point implies $\psi$ is zero everywhere $\square$

*Remark* 5.5. In the above proof, while taking $\psi \in ker\not{D}^+$ some confusion may arise if $ker\not{D}^+ \subset \Gamma(\mathbb{S}_+)$ is not specified. A reader might think that $\not{D}^+$ acts on $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$ and $\psi$ is equal to something like $(\psi, \eta)$ for some nonzero $\eta$, so that $\not{D}^2\psi = \not{D}^+\not{D}^-\psi$.

Alternatively, we could use the Weitzenböck Formula for the Hodge/modern Laplacian to show that there are no PSC anti-self-dual(ASD) metrics on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This is a weaker conclusion though sufficient for our purposes

**Theorem 5.6** (Weitzenböck Formula 2[LeOM])**.** *On a Riemannian manifold, we can express the Hodge/modern Laplacian in terms of the connection/rough Laplacian as*

$$(d + d^*)^2 = \nabla^*\nabla - 2W + \frac{s}{3}$$

*where $\nabla$ is the Riemannian connection and $W$ is the Weyl curvature tensor.*

**Theorem 5.7.** *The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any anti-self-dual(ASD) metric of positive scalar curvature(PSC).*

*Proof.* Again we are going to show this only for $K3$ as in (5.4). Suppose we have a metric of positive scalar curvature.

Anti-self-duality reduces our Weitzenbock Formula (5.6) to the form

$$(d+d^*)^2 = \nabla^*\nabla - 2W_- + \frac{s}{3}$$

because $W = W_-$ or $W_+ = 0$.

We have already explained in (4.2) that $b_2^+$ of the $K3$ surface is nonzero. So take a nontrivial harmonic self-dual 2-form $\varphi$. $W_- : \Gamma(\Lambda^-) \to \Gamma(\Lambda^-)$ only acts on anti-self-dual forms, so it takes $\varphi$ to zero. Applying the formula,

$$0 = \nabla^*\nabla\varphi + \frac{s}{3}\varphi$$

and taking the inner product with $\varphi$ and integrating over the manifold yields similarly

$$0 = (\nabla^*\nabla\varphi, \varphi)_{L^2} + (\frac{s}{3}\varphi, \varphi)_{L^2} = (\nabla\varphi, \nabla\varphi)_{L^2} + \frac{s}{3}(\varphi, \varphi)_{L^2} = \int_M (|\nabla\varphi|^2 + \frac{s}{3}|\varphi|^2)dV$$

and $s > 0$ implies that $|\nabla\varphi| = |\varphi| = 0$ everywhere, hence $\varphi \equiv 0$, a contradiction. □

## 6. Other Examples

In this section, we will go through some examples. We begin with the case $b^+ = 1$.

**Theorem 6.1** ([KimLePon],[RS-SFK])**.** *For all integers $k \geq 10$, the connected sum $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ admits scalar-flat-Kähler(SFK) metrics.*[3]

The case $k \geq 14$ is achieved in [KimLePon]. They start with blow ups of $\mathbb{CP}_1 \times \Sigma_2$ the cartesian product of rational curve and genus-2 curve, which already have a SFK metric via the hyperbolic ansatz of [LeExp]. After applying an isometric involution, they get a SFK orbifold, which has isolated singularities modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Replacing these singular models with smooth ones, they obtain the desired metric.

For the case $k \geq 10$, Rollin and Singer first construct a related SFK orbifold with isolated and cyclic singularities of which the algebra $\mathfrak{a}_0$ of non-parallel holomorphic vector fields is zero. This is done by an argument analogous to that of [Burns-Bart]. The target manifold is the minimal resolution of this orbifold. To obtain the target metric, they glue some suitable local models of SFK metrics to the orbifold. These models are asymptotically locally Euclidean(ALE) scalar flat Kähler metrics constructed in [Cal-Sing].

When a metric is Kähler, from the decomposition of the Riemann Curvature operator, scalar-flatness turns out to be equivalent to being anti-self-dual. So these metrics are SF-ASD.

Since these manifolds have $b^+ \neq 0$ Weitzenböck Formulas apply as in section §5, so automatically the scalar curvature can not change sign. These examples show why the case $b^+ = 0$ we focussed on, is interesting.

A second type of example is [4]

---

[3]It is a curious fact that $k = 10$ is the minimal number for these type of metrics(SF-ASD) on $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, known by [LeSD] long before these constructions made. See [LeOM] for a survey.

[4]Thanks to the first referee for pointing out this example and the remark.

*Example* 6.2. Let $\Sigma_g$ be the genus-$g(>1)$ surface with Kähler metric of constant curvature $\kappa = -1$, and $S^2$ be the 2-sphere with the round $\kappa = +1$ metric. Consider the product metric on $S^2 \times \Sigma_g$ which is Kähler with zero scalar curvature. So this metric is anti-self-dual. Then we have fixed point free, orientation reversing, isometric involutions of both surfaces obtained by antipodal maps. Combination of these involutions yield an isometric involution on the product and the metric pushes down to a metric on

$$(S^2 \times \Sigma_g)/\mathbb{Z}_2 = \mathbb{R}P^2 \times (\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{(g+1)-many})$$

which is SF-ASD as these properties are preserved under isometry. This is an example with all the key properties $(b^+ = 0)$where the metric is completely explicit. Note that this manifold is non-orientable.

One has to be careful about the involution on $\Sigma_g$ though. There are many hyperbolic metrics on $\Sigma_g$ which do not have an isometric involution satisfying our conditions. Involution must be conformal. One way to achieve this is as follows. We take a conformal structure on the $(g+1)\mathbb{R}P^2$, and pull this back to its orientable double cover $\Sigma_g$. By the uniformization theorem of Riemann Surfaces, there is a unique hyperbolic$(\kappa = -1)$ metric of $\Sigma_g$ in this conformal class. Since this metric is unique in its conformal class, it is automatically invariant under the involution and pushes down to a hyperbolic metric on $(g+1)\mathbb{R}P^2$. It is known that the moduli space of hyperbolic metrics on $\Sigma_g$ is $6g-6$ real dimensional, on the other hand $3g-3$ real dimensional on $(g+1)\mathbb{R}P^2$. So it is appearent that there are many hyperbolic metrics on $\Sigma_g$ which are not coming from the quotient. So that they do not have the isometric involution of the kind we use.

Another way to construct this involution can be to begin with a surface in $\mathbb{R}^3$ which is symmetric about the origin, e.g. add symmetric handles to a sphere or a torus about the origin. Then take the conformal structure induced from Euclidean $\mathbb{R}^3$. There is a unique hyperbolic metric that induces this conformal structure, so proceed as before.

*Remark* 6.3. The other side of the story discussed here is that we have $ASD$, conformally flat deformations to negative scalar curvature metrics e.g. on $M = \Sigma_g \times S^2$. It is obtained by deforming

$$\rho : \pi_1(M) \longrightarrow SL(2, \mathbb{H}),$$

the representation of the fundamental group in $SL(2, \mathbb{H})$. This is the group of conformal transformations of $S^4$. It contains the isometry group $SL(2, \mathbb{R})$ due to the fact that $\mathcal{H}^2 \times S^2$ is conformally flat and conformally diffeomorphic to $S^4 - S^1$. This is the universal cover of $\Sigma_g \times S^2$, and its fundamental group acts by conformal transformations in $SL(2, \mathbb{R}) \subset SL(2, \mathbb{H})$. $\pi_1(M) = \pi_1(\Sigma_g)$ is generated by $2g$ elements $\{a_1, b_1 \ \cdots \ a_g, b_g\}$ and these are subject to the single relation $\prod [a_j, b_j] = 1$ the product of the commutators. So a representation in $SL(2, \mathbb{H})$ corresponds to a choice of $2g$ elements and a relation. Since this Lie group is 15 dimensional and we have to subtract the change of basis conjugation and the relations , this kind of representations depend on $15 \times 2g - 15 \times 1 - 15 \times 1 = 30g - 30$ parameters. On the other hand, a twisted product metric on $\Sigma_g \times S^2$ provides a representation in $SL(2,\mathbb{R}) \times SO(3)$, which is a $3+3=6$ dimensional Lie group. So we have $6 \times 2g - 6 \times 1 - 6 \times 1 = 12g - 12$ parameters for this representation. This difference means that the generic conformally flat structure on $M$ does not come from a twisted product metric. We refer [Pon] for further details.

Using the Weitzenböck Formula (5.6), LeBrun [LeSD] shows that a conformally flat metric on $M$ of zero scalar curvature must be Kähler with respect to both orientations, and by a holonomy argument he further shows that the metric is of twisted product type.[5]

Similar parameter counts are valid for $M/\mathbb{Z}_2$ and this shows that the generic conformally flat metric on this manifold has negative Yamabe constant.

Also, by further investigation, it might be possible to get examples which are doubly covered by e.g. the simply connected examples of [KimLePon].

**Acknowledgments.** I want to thank to Claude LeBrun for his excellent directions, to Ioana Suvaina for useful discussions, and to both of the referees for excellent comments, corrections and suggestions. I would like to express my gratitude to the organizers of the Gökova Geometry-Topology Conference, especially to Turgut Önder, Selman Akbulut and Sema Salur for providing such an academically élite and beautiful scientific atmosphere each year.

## References

[AHS] M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A, 362 (1978), pp. 425–461.

[BPV] W. Barth, C. Peters, and A. V. de Ven, *Compact Complex Surfaces*, Springer-Verlag, 1984.

[Beauville] A. Beauville, Complex algebraic surfaces, 2nd edition. London Mathematical Society Student Texts, 34.Cambridge University Press, (1996).

[Bes] A. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

[Burns-Bart] D. Burns and P. de Bartolomeis, *Stability of vector bundles and extremal metrics* Invent. Math. 92, 403–407 (1988)

[Ca] E. Calabi, *The space of Kähler metrics*, In Proceedings of the ICM, Amsterdam 1954 vol 2 pp. 206–207 North-Holland, Amsterdam 1956

[Cal-Sing] D. Calderbank and M. Singer, *Einstein metrics and complex singularities*, Invent. Math. 156, 405–443 (2004)

[DonFried] S. Donaldson and R. Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity, 2 (1989), pp. 197–239.

[Fr] T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics, 25. American Mathematical Society, Providence, RI, 2000

[GS] R. Gompf and A. Stipsicz, 4-*Manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999

[GHJ] M. Gross, D. Huybrechts, D. Joyce, *Calabi-Yau manifolds and related geometries*, Lectures from the Summer School held in Nordfjordeid, June 2001. Universitext. Springer-Verlag, Berlin, 2003

[HitEin] N. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Differential Geometry 9 (1974), 435–441

[5] Thanks to C.LeBrun for this discussion.

[Joyce] D. JOYCE, *Compact manifolds with special holonomy*, Oxford Mathematical Monographs. Oxford University Press 2000

[Kalafat] M. KALAFAT, *Scalar Curvature and Connected Sums Of Self-Dual 4-Manifolds*, available on the Arxiv math.DG/0611769

[Kim] JONGSU KIM, *On the scalar curvature of self-dual manifolds*, Math. Ann. 297 (1993), no. 2, 235–251

[KimLePon] J. KIM, C. LEBRUN, M. PONTECORVO, *Scalar-flat Kähler surfaces of all genera*, J. Reine Angew. Math. 486 (1997), 69–95

[KN1] S. KOBAYASHI AND K. NOMIZU, *Foundations of differential geometry 1*, Wiley & Sons, New York 1963, 1969

[LM] H.B. LAWSON AND M.L. MICHELSOHN, *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989

[LeOM] C. LEBRUN,*Curvature Functionals, Optimal Metrics, and the Differential Topology of 4-Manifolds*, in Different Faces of Geometry, Donaldson, Eliashberg, and Gromov, editors, Kluwer Academic/Plenum, 2004.

[LeExp] C. LEBRUN, Explicit self-dual metrics on $C\mathrm{P}_2\# \cdots \# C\mathrm{P}_2$ J. Differential Geom. 34 (1991), no. 1, 223–253

[LeSD] C. LEBRUN, On the Topology of Self-Dual 4-Manifolds, Proc. Am. Math. Soc. 98 (1986) 637–640

[LP] J. LEE AND T. PARKER, *The Yamabe problem*, Bull. Am. Math. Soc., 17 (1987), pp. 37–91.

[McI] BRETT MCINNES, *Methods of holonomy theory for Ricci-flat Riemannian manifolds*, J. Math. Phys. 32 (1991), no. 4, 888–896

[MoSW] J.W. MORGAN, *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, Mathematical Notes, 44. Princeton University Press, Princeton, NJ, 1996

[Pet] P. PETERSEN, *Riemannian geometry*, Graduate Texts in Mathematics, 171. Springer-Verlag, New York, 1998

[Pon] M. PONTECORVO, *Uniformization of conformally flat Hermitian surfaces*, Differential Geom. Appl.2 (1992), no. 3, 295–305

[Poon86] Y.S. POON, *Compact self-dual manifolds with positive scalar curvature*, J. Differential Geom. 24 (1986), no. 1, 97–132

[PrWe] M. H. PROTTER AND H. F. WEINBERGER, *Maximum principles in differential equations*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.

[RS-SFK] Y. ROLLIN, M. SINGER, *Non-minimal scalar-flat Kähler surfaces and parabolic stability*, Invent. Math. 162 (2005), no. 2, 235–270

[Taubes] C.H. TAUBES, *The existence of anti-self-dual conformal structures*, J. Differential Geom. 36 (1992), no. 1, 163–253.

[War] F. WARNER, *Foundations of Differentiable Manifolds and Lie Groups*, Springer-Verlag 1983

[Yau] S.T. YAU, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation I*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, STONYBROOK, NY 11794, USA
*E-mail address*: kalafat@math.sunysb.edu