# Scalar Curvature and Connected Sums of Self-Dual 4-Manifolds

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**Abstract**

In [DF] under a reasonable vanishing hypothesis, Donaldson and Friedman proved that the connected sum of two self-dual Riemannian 4-Manifolds is again self-dual. Here we prove that the same result can be extended over to the positive scalar curvature case.

The proof is based on the twistor theory. First a technical vanishing theorem is established by using an appropriate spectral sequence, then Green's functions and Serre-Horrocks vector bundle constructions are used to detect the sign of the scalar curvature.

## 1 Introduction

Let $(M, g)$ be an oriented Riemannian n-manifold. Then by raising an index, the Riemann curvature tensor at any point can be viewed as an operator $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ hence an element of $S^2\Lambda^2 M$. It satisfies the algebraic Bianchi identity hence lies in the vector space of *algebraic curvature tensors*. This space is an $O(n)$-module and has an orthogonal decomposition into irreducible subspaces for $n \geq 4$. Accordingly the Riemann curvature operator decomposes as:

$$\mathcal{R} = U \oplus Z \oplus W$$

where

$$U = \frac{s}{2n(n-1)} g \bullet g \quad \text{and} \quad Z = \frac{1}{n-2} \overset{\circ}{Ric} \bullet g$$

$s$ is the scalar curvature, $\overset{\circ}{Ric} = Ric - \frac{s}{n} g$ is the trace-free Ricci tensor, "$\bullet$" is the Kulkarni-Nomizu product, and $W$ is the *Weyl Tensor* which is defined to be what is left over from the first two pieces.

When we restrict ourselves to dimension $n = 4$, the Hodge Star operator $* : \Lambda^2 \to \Lambda^2$ is an involution and has $\pm 1$-eigenspaces decomposing the space of two forms as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, yielding a decomposition of any operator acting on this space. In particular $W_\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ is called the self-dual and anti-self-dual pieces of the Weyl curvature operator. And we call $g$ to be *self-dual*(resp. *anti-self-dual*) *metric* if $W_-$(resp. $W_+$) vanishes. In this case [AHS] construct a complex 3-manifold $Z$ called the *Twistor Space* of $(M^4, g)$, which comes with a fibration by holomorphically embedded rational curves :

$$\begin{array}{ccccc} \mathbb{CP}_1 & \to & Z & & \text{Complex 3-manifold} \\ & & \downarrow & & \\ & & M^4 & & \text{Riemannian 4-manifold} \end{array}$$

This construction drew the attention of geometers, and many examples of Self-Dual metrics and related Twistor spaces were given afterwards. One result proved to be a quite effective way to produce infinitely many examples and became a cornerstone in the field :

**Theorem 2.1** (Donaldson-Friedman,1989,[DF]). *If* $(M_1, g_1)$ *and* $(M_2, g_2)$ *are compact self-dual Riemannian 4-manifolds with* $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$
*Then* $M_1 \# M_2$ *also admits a self-dual metric.*

The idea of the proof is to work upstairs in the complex category rather than downstairs. One glues the blown up twistor spaces from their exceptional divisors to obtain a singular complex space $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$. Then using the Kodaira-Spencer deformation theory extended by R.Friedman to singular spaces, one obtains a smooth complex manifold, which turns out to be the twistor space of the connected sum.

When working in differential geometry, one often deals with the moduli space of certain kind of metrics. The situation is also the same for the self-dual theory. Many people obtained results on the space of positive scalar curvature self-dual(PSC-SD) metrics on various kinds of manifolds. Since the positivity of the scalar curvature imposes some topological restrictions on the moduli space, people often find it convenient to work under this assumption.

However one realizes that there is no connected sum theorem for self-dual positive scalar curvature metrics. Donaldson-Friedman Theorem(2.1) does not make any statement about the scalar curvature of the metrics produced. Therefore we attacked the problem of determining the sign of the scalar curvature for the metrics produced over the connected sum, beginning by proving the following, using the techniques similar to that of [LeOM]:

**Theorem 5.3** (Vanishing Theorem). *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-parameter standard deformation of $Z_0$, where $Z_0$ is as in Theorem (2.1), and $\mathcal{U} \subset \mathbb{C}$ is a neighborhood of the origin. Let $L \to \mathcal{Z}$ be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\tilde{Z}_1}(K_{\mathcal{Z}}^{1/2}).$$

*If $(M_i, [g_i])$ has positive scalar curvature, then by possibly replacing $\mathcal{U}$ with a smaller neighborhood of $0 \in \mathbb{C}$ and simultaneously replacing $\mathcal{Z}$ with its inverse image, we can arrange for our complex 4-fold $\mathcal{Z}$ to satisfy*

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

The proof makes use of the Leray Spectral Sequence, homological algebra and Kodaira-Spencer deformation theory, involving many steps. Using this technical theorem next we prove that the Donaldson-Friedman Theorem can be generalized to the positive scalar curvature(PSC) case :

**Theorem 8.1.** *Let $(M_1, g_1)$ and $(M_2, g_2)$be compact self-dual Riemannian 4-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature.*

*Then, for all sufficiently small $\mathfrak{t} > 0$, the self-dual conformal class $[g_{\mathfrak{t}}]$ obtained on $M_1 \# M_2$ by the Donaldson-Friedman Theorem (2.1) contains a metric of positive scalar curvature.*

We work on the self-dual conformal classes constructed by the Donaldson-Friedman Theorem (2.1). Conformal Green's Functions[LeOM] are used to detect the sign of the scalar curvature of these metrics. Positivity for the scalar curvature is characterized by non-triviality of the Green's Functions. Then the Vanishing Theorem (5.3) will provide the Serre-Horrocks[Ser, Hor] vector bundle construction, which gives the Serre Class, a substitute for the Green's Function by Atiyah[AtGr]. And non-triviality of the Serre Class will provide the non-triviality of the extension described by it.

In sections §2-§4 we review the background material. In §5 the vanishing theorem is proven, and finally in §6-§8 the sign of the scalar curvature is detected.

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## 2 Self-Dual Manifolds and the Donaldson-Friedman Construction

One of the main improvements in the field of self-dual Riemannian 4-manifolds is the connected sum theorem of Donaldson and Friedman [DF] published in 1989. If $M_1$ and $M_2$ admit self-dual metrics, then under certain circumstances their connected sum admits, too . This helped us to create many examples of self-dual manifolds. If we state it more precisely :

**Theorem 2.1** (Donaldson-Friedman[DF]). *Let $(M_1, g_1)$ and $(M_2, g_2)$ be compact self-dual Riemannian 4-manifolds and $Z_i$ denote the corresponding twistor spaces. Suppose that $H^2(Z_i, \mathcal{O}(TZ_i)) = 0 \quad for \quad i = 1, 2$.*

*Then, there are self-dual conformal classes on $M_1 \# M_2$ whose twistor spaces arise as fibers in a 1-parameter standard deformation of $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$.*

We devote the rest of this section to understand the statement and the ideas in the proof of this theorem since our main result (8.1) is going to be a generalization of this celebrated theorem.

The idea is to work upstairs in the complex category rather than downstairs. So let $p_i \in M_i$ be arbitrary points in the manifolds. Consider their inverse images $C_i \approx \mathbb{CP}_1$ under the twistor fibration, which are twistor lines, i.e. rational curves invariant under the involution. Blow up the twistor spaces $Z_i$ along these rational curves. Denote the exceptional divisors by $Q_i \approx \mathbb{CP}_1 \times \mathbb{CP}_1$ and the blown up twistor spaces by $\widetilde{Z}_i = Bl(Z_i, C_i)$ . The normal bundles for the exceptional divisors is computed by :

**Lemma 2.2** (Normal Bundle). *The normal bundle of $Q_2$ in $\widetilde{Z}_2$ is computed to be*

$$NQ_2 = N_{Q_2/\widetilde{Z}_2} \approx \mathcal{O}(1, -1)$$

*where the second component is the fiber direction in the blowing up process.*

*Proof.* We split the computation into the following steps

1. We know that $N_{C_2/Z_2} \approx \mathcal{O}(1) \oplus \mathcal{O}(1)$ and we compute its second wedge power as

$$c_1(\wedge^2\mathcal{O}(1)\oplus\mathcal{O}(1))[\mathbb{P}_1] = c_1(\mathcal{O}(1)\oplus\mathcal{O}(1))[\mathbb{P}_1] = (c_1\mathcal{O}(1)+c_1\mathcal{O}(1))[\mathbb{P}_1] = 2$$

by the Whitney product identity of the characteristic classes. so we have

$$\wedge^2 N_{C_2/Z_2} \approx \mathcal{O}_{\mathbb{P}_1}(2)$$

2. $K_Q = \pi_1^* K_{\mathbb{P}_1} \otimes \pi_2^* K_{\mathbb{P}_1} = \pi_1^*\mathcal{O}(-2) \otimes \pi_2^*\mathcal{O}(-2) = \mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_1}(-2,-2)$

3. $K_Q = K_{\widetilde{Z}_2} + Q|_Q = \pi^* K_{Z_2} + 2Q|_Q = \pi^*(K_{Z_2}|_{\mathbb{P}_1}) + 2Q|_Q = \pi^*(K_{\mathbb{P}_1} \otimes \wedge^2 N^*_{\mathbb{P}_1/Z_2}) + 2Q|_Q = \pi^*(\mathcal{O}(-2)\otimes\mathcal{O}(-2)) + 2Q|_Q = \pi^*\mathcal{O}(-4) + 2Q|_Q$
since the second component is the fiber direction, the pullback bundle will be trivial on that so $\pi^*\mathcal{O}(-4) = \mathcal{O}(-4,0)$ solving for $Q|_Q$ now gives us

$$N_{Q/\widetilde{Z}_2} = Q|_Q = (K_Q \otimes \pi^*\mathcal{O}(-4)^*)^{1/2} = (\mathcal{O}(-2,-2)\otimes\mathcal{O}(4,0))^{1/2} = \mathcal{O}(1,-1)$$

□

We then construct the complex analytic space $Z_0$ by identifying $Q_1$ and $Q_2$ so that it has a normal crossing singularity

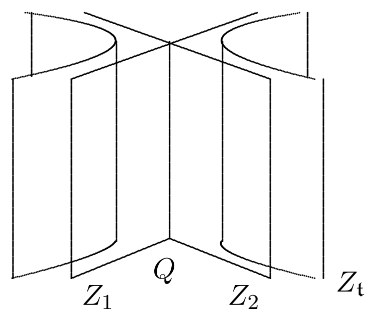
$$Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2.$$

Carrying out this identification needs a little bit of care. We interchange the components of $\mathbb{CP}_1 \times \mathbb{CP}_1$ in the gluing process so that the normal bundles $N_{Q_1/\widetilde{Z}_1}$ and $N_{Q_2/\widetilde{Z}_2}$ are dual to each other. Moreover we should respect to the real structures. The real structures $\sigma_1$ and $\sigma_2$ must agree on $Q$ obtained by identifying $Q_1$ with $Q_2$, so that the real structures extend over $Z_0$ and form the anti-holomorphic involution $\sigma_0 : Z_0 \to Z_0$.

Now we will be trying to deform the singular space $Z_0$, for which the Kodaira-Spencer's standard deformation theory does not work since it is only for manifolds it does not tell anything about the deformations of the singular spaces. We must use the theory of deformations of a compact reduced complex analytic spaces, which is provided by R.[Friedman]. This generalized theory is quite parallel to the theory of manifolds. The basic modification is that the roles of $H^i(\Theta)$ are now taken up by the groups $T^i = \mathrm{Ext}^i(\Omega^1, \mathcal{O})$.

We have assumed $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ so that the deformations of $Z_i$ are unobstructed. Donaldson and Friedman are able to show that $T^2_{Z_0} = \mathrm{Ext}^2_{Z_0}(\Omega^1, \mathcal{O}) = 0$ so the deformations of the singular space is unobstructed. We have a versal family of deformations of $Z_0$. This family is parameterized by a neighborhood of the the origin in $\mathrm{Ext}^1_{Z_0}(\Omega^1, \mathcal{O})$. The generic fiber is non-singular and the real structure $\sigma_0$ extends to the total space of this family.

$$\begin{aligned} \omega :\quad \mathcal{Z} &\to \mathcal{U} \qquad \text{for} \qquad Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2 \\ Z_0 &\longmapsto 0 \end{aligned}$$




Instead of working with the entire versal family, it is convenient to work with certain subfamilies, called *standard deformations*:

**Definition 2.3** ([LeOM]). *A* 1-parameter standard deformation *of $Z_0$ is a flat proper holomorphic map $\omega : \mathcal{Z} \to \mathcal{U} \subset \mathbb{C}$ of a complex 4-manifold to an open neighborhood of $0$, together with an anti-holomorphic involution $\sigma : \mathcal{Z} \to \mathcal{Z}$, such that*

- $\omega^{-1}(0) = Z_0$
- $\sigma|_{Z_0} = \sigma_0$
- *$\sigma$ descents to the complex conjugation in $\mathcal{U}$*
- *$\omega$ is a submersion away from $Q \subset Z_0$*
- *$\omega$ is modeled by $(x, y, z, w) \mapsto xy$ near any point of $Q$.*

We also define

**Definition 2.4** (Flat Map[H]). *Let $K$ be module over a ring $A$. We say that $K$ is* flat *over $A$ if the functor $L \mapsto K \otimes_A L$ is an exact functor for all modules $L$ over $A$.*
*Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is* flat *over $Y$ if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module for any $x$. Where $y = f(x)$, $\mathcal{F}_x$ is considered to be a $\mathcal{O}_{y,Y}$-module via the natural map $f^\# : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. We say $X$ is* flat *over $Y$ if $\mathcal{O}_X$ is.*

Then for sufficiently small, nonzero, real $\mathfrak{t} \in \mathcal{U}$ the complex space $Z_\mathfrak{t} = \omega^{-1}(\mathfrak{t})$ is smooth and one can show that it is the twistor space of a self-dual metric on $M_1 \# M_2$.

# 3 The Leray Spectral Sequence

Given a continuous map $f : X \to Y$ between topological spaces, and a sheaf $\mathcal{F}$ over $X$, the *$q$-th direct image sheaf* is the sheaf $R^q(f_*\mathcal{F})$ on $Y$ associated to the presheaf $V \to H^q(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. This is actually the right derived functor of the functor $f_*$. The Leray Spectral Sequence is a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(Y, R^q(f_*\mathcal{F}))$$

$$E_\infty^{p,q} = H^{p+q}(X, \mathcal{F})$$

The first page of this spectral sequence reads :

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
 & H^0(Y, R^2(f_*\mathcal{F})) & H^1(Y, R^2(f_*\mathcal{F})) & H^2(Y, R^2(f_*\mathcal{F})) & \cdots \\
 & H^0(Y, R^1(f_*\mathcal{F})) & H^1(Y, R^1(f_*\mathcal{F})) & H^2(Y, R^1(f_*\mathcal{F})) & \cdots \\
E_2 & H^0(Y, R^0(f_*\mathcal{F})) & H^1(Y, R^0(f_*\mathcal{F})) & H^2(Y, R^0(f_*\mathcal{F})) & \cdots \\
\hline
\end{array}$$

A degenerate case is when $R^i(f_*\mathcal{F}) = 0$ for all $i > 0$.

**Remark 3.1.** *This is the case if $\mathcal{F}$ is flabby for example. Remember that to be* flabby *means that the restriction map $r : \mathcal{F}(B) \to \mathcal{F}(A)$ is onto for open sets $B \subset A$. In this case $H^i(X, \mathcal{F}) = 0$ for $i > 0$ as well as $H^i(U, \mathcal{F}|_U) = 0$ for $U$ open, because the restriction of a flabby sheaf to any open subset is again flabby by definition. That means $H^q(f^{-1}(.), \mathcal{F}|_.) = 0$ for all $q > 0$ so $R^i(f_*\mathcal{F}) = 0$.*

When the spectral sequence degenerates this way, the second and succeeding rows of the first page vanish. And because $V \to H^0(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ is the presheaf of the direct image sheaf, we have $R^0 f_* = f_*$. So the first row consist of $H^i(Y, f_*\mathcal{F})$'s. Vanishing of the differentials cause immediate convergence to $E_\infty^{i,0} = H^{i+0}(X, \mathcal{F})$. So we got:

**Proposition 3.2.** *If $R^i(f_*\mathcal{F}) = 0$ for all $i > 0$, then $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ naturally for all $i \geq 0$.*

As another example, the following proposition reveals a different sufficient condition for this degeneration. See [Voisin] v2 , p124 for a sketch of the proof:

**Proposition 3.3** (Small Fiber Theorem)**.** *Let $f : X \to Y$ be a holomorphic, proper and submersive map between complex manifolds, $\mathcal{F}$ a coherent analytic sheaf or a holomorphic vector bundle on $X$. Then $H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = 0$ for all $y \in Y$ implies that $R^i(f_*\mathcal{F}) = 0$.*

As an application of these two propositions, we obtain the main result of this section :

**Proposition 3.4.** *Let $Z$ be a complex $n$-manifold with a complex $k$-dimensional submanifold $V$. Let $\widetilde{Z}$ denote the blow up of $Z$ along $V$, with blow up map $\pi : \widetilde{Z} \to Z$. Let $\mathcal{G}$ denote a coherent analytic sheaf(or a vector bundle) over $Z$. Then we can compute the cohomology of $\mathcal{G}$ on either side i.e.*

$$H^i(\widetilde{Z}, \pi^*\mathcal{G}) = H^i(Z, \mathcal{G}).$$

*Proof.* The inverse image of a generic point on $Z$ is a point, else a $\mathbb{P}^{n-k-1}$. We have

$$H^i(f^{-1}(y), \pi^*\mathcal{G}|f^{-1}(y)) = H^i(\mathbb{P}^{n-k-1}, \mathcal{O}) = H_{\bar\partial}^{0,i}(\mathbb{P}^{n-k-1}) = 0$$

at most, since the cohomology of $\mathbb{P}^{n-k-1}$ is accumulated in the middle for $i > 0$. So that we can apply Proposition (3.3) to get $R^i(\pi_*\pi^*\mathcal{G}) = 0$ for all $i > 0$. Which is the hypothesis of Proposition (3.2), so we get $H^i(\widetilde{Z}, \pi^*\mathcal{G}) = H^i(Z, \pi_*\pi^*\mathcal{G})$ naturally for all $i \geq 0$, and the latter equals $H^i(Z, \mathcal{G})$ since $\pi_*\pi^*\mathcal{G} = \mathcal{G}$ by the combination of the following two lemmas. □

**Lemma 3.5** (Projection Formula[H])**.** *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank. Then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E},$$

*in particular for* $\mathcal{F} = \mathcal{O}_X$

$$f_* f^* \mathcal{E} = f_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

**Lemma 3.6** (Zariski's Main Theorem,Weak Version[H])**.** *Let* $f : X \to Y$ *be a birational projective morphism of noetherian integral schemes, and assume that* $Y$ *is normal. Then* $f_* \mathcal{O}_X = \mathcal{O}_Y$*.*

*Proof.* The question is local on $Y$. So we may assume that $Y$is affine and equal to $SpecA$. Then $f_* \mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, so $B = \Gamma(Y, f_* \mathcal{O}_X)$ is a finitely generated $A$-module. But $A$ and $B$ are integral domains with the same quotient field, and $A$ is integrally closed, we must have $A = B$. Thus $f_* \mathcal{O}_X = \mathcal{O}_Y$ □

# 4 Natural square root of the canonical bundle of a twistor space

In the next section, we are going to prove that a certain cohomology group of a line bundle vanishes. For that we need some lemmas. First of all, the canonical bundle of a twistor space $Z$ has a natural square root, equivalently $Z$ is a spin manifold as follows:

The Riemannian connection of $M$ acts on the 2-forms hence the twistor space, accordingly we can split the tangent bundle $T_x Z = T_x F \oplus (p^* TM)_x$. The complex structure on $(p^* TM)_x$ is obtained from the identification $\cdot\varphi : T_x M \longleftrightarrow (\mathbb{V}_+)_x$ provided by the Clifford multiplication of a non-zero spinor $\varphi \in (\mathbb{V}_+)_x$. This identification is linear in $\varphi$ as $\varphi$ varies over $(\mathbb{V}_+)_x$. So we have a nonvanishing section of $\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}\mathbb{V}_-}(1)$ with values in $Hom(TM, \mathbb{V}_+)$ or $Hom(p^*TM, p^*\mathbb{V}_+)$ trivializing the bundle

$$\mathcal{O}_Z(1) \otimes Hom(p^*TM, p^*\mathbb{V}_+) = \mathcal{O}_Z(1) \otimes p^*TM^* \otimes p^*\mathbb{V}_+ \approx p^*TM^* \otimes \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+$$

hence yielding a natural isomorphism

$$p^*TM \approx \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+, \tag{1}$$

where $\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}\mathbb{V}_-}(1)$ is the positive Hopf bundle on the fiber.

The Hopf bundle exist locally in general, so as the isomorphism. If $M$ is a spin manifold $\mathbb{V}_\pm$ exist globally on $M$ and $\mathcal{O}_Z(1)$ exist on $Z$, so our isomorphism holds globally. Furthermore, we have a second isomorphism holding for any projective bundle, obtained as follows (see [Fulton] p434 , [Zheng] p108) :

Let E be a complex vector bundle of rank-$(n+1)$ over $M$, $p : \mathbb{P}E \to M$ its projectivization. We have the imbedding of the tautological line bundle $\mathcal{O}_{\mathbb{P}E}(-1) \hookrightarrow p^*E$. Giving the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*E \to p^*E/\mathcal{O}_{\mathbb{P}E}(-1) \to 0,$$

tensoring by $\mathcal{O}_{\mathbb{P}E}(1)$ gives

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \mathcal{O}_{\mathbb{P}E}(1) \otimes p^*E \to T_{\mathbb{P}E/M} \to 0$$

where $T_{\mathbb{P}E/M} \approx Hom(\mathcal{O}(-1), \mathcal{O}(-1)^{\perp}) = Hom(\mathcal{O}(-1), p^*E/\mathcal{O}(-1)) = \mathcal{O}(1) \otimes p^*E/\mathcal{O}(-1)$ is the relative tangent bundle of $\mathbb{P}E$ over $M$, originally defined to be ${\Omega^1_{\mathbb{P}E/M}}^*$. Taking $E = \mathbb{V}_-$ , $TF$ denoting the tangent bundle over the fibers:

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_- \to TF \to 0$$

so, we got our second isomorphism :

$$TF \oplus \mathcal{O}_Z \approx \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_- \tag{2}$$

Now we are going to compute the first chern class of the spin bundles $\mathbb{V}_{\pm}$, and see that $c_1(\mathbb{V}_{\pm}) = 0$. Choose a connection $\nabla$ on $\mathbb{V}_{\pm}$. Following [KN] it defines a connection on the associated principal $\mathfrak{su}(2)$ bundle $P$, with connection one form $\omega \in A^1(P, \mathfrak{su}(2))$ defined by the projection [Morita] $T_uP \to V_u \approx \mathfrak{su}(2)$ having curvature two form $\Omega \in A^2(P, \mathfrak{su}(2))$ defined by [KN] :

$$\Omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)] \text{ for } X, Y \in T_uP.$$

We define the first polynomial functions $f_0, f_1, f_2$ on the lie algebra $\mathfrak{su}(2)$ by

$$det(\lambda I_2 + \frac{i}{2\pi}M) = \sum_{k=0}^{2} f_{2-k}(M)\lambda^k = f_0(M)\lambda^2 + f_1(M)\lambda + f_2(M) \text{ for } M \in \mathfrak{su}(2).$$

Then these polynomials $f_i : \mathfrak{su}(2) \to \mathbb{R}$ are invariant under the adjoint action of $SU(2)$, denoted $f_i \in I^1(SU(2))$, namely

$$f_i(\mathrm{ad}_g(M)) = f_i(M) \text{ for } g \in SU(2) \text{ , } M \in \mathfrak{su}(2)$$

where $\mathrm{ad}_g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is defined by $\mathrm{ad}_g(M) = R_{g^{-1}*}L_{g*}(M)$.
If we apply any $f \in I^1(SU(2))$ after $\Omega$ we obtain:

$$f \circ \Omega \ : \ T_uP \times T_uP \longrightarrow \mathfrak{su}(2) \longrightarrow \mathbb{R}.$$

It turns out that $f \circ \Omega$ is a closed form and projects to a unique 2-form say $\overline{f \circ \Omega}$ on $M$ i.e. $f \circ \Omega = \pi^*(\overline{f \circ \Omega})$ where $\pi : P \to M$. By the way, a $q$-form $\varphi$ on $P$ projects to a unique $q$-form, say $\overline{\varphi}$ on $M$ if $\varphi(X_1 \cdot \cdot X_q) = 0$ whenever at least one of the $X_i$'s is vertical and $\varphi(R_{g*}X_1 \cdot \cdot R_{g*}X_q) = \varphi(X_1 \cdot \cdot X_q)$. $\overline{\varphi}$ on $M$ defined by $\overline{\varphi}(V_1 \cdot \cdot Vq) = \varphi(X_1 \cdot \cdot X_q), \pi(X_i) = V_i$ is independent of the choice of $X_i$'s. See [KN]v2p294 for details.
So, composing with $\Omega$ and projecting defines a map w : $I^1(SU(2)) \to H^2(M, \mathbb{R})$ called the Weil homomorphism, it is actually an algebra homomorphism when extended to the other gradings.
Finally, the chern classes are defined by $c_k(\mathbb{V}_\pm) := \left[\overline{f_k \circ \Omega}\right]$ independent of the connection chosen. Notice that $f_2(M) = det(\frac{i}{2\pi}M), f_1(M) = tr(\frac{i}{2\pi}M)$ in our case. And if $M \in \mathfrak{su}(2)$ then $e^M \in SU(2)$ implying $1 = det(e^M) = e^{trM}$ and $trM = 0$. But $\Omega$ is of valued $\mathfrak{su}(2)$, so if you apply the $f_1 = tr$ after $\Omega$ you get 0. Causing $c_1(\mathbb{V}_\pm) = \left[\overline{tr(\frac{i}{2\pi}\Omega)}\right] = 0$.
One last remark is that $\overline{f_k \circ \Omega} = \gamma_k$ in the notation of [KN], $\gamma_1 = P^1(\frac{i}{2\pi}\Theta) = tr(\frac{i}{2\pi}\Theta)$ in the notation of [GH]p141,p407. And $\Omega = \pi^*\Theta$ in the line bundle case.

Vanishing of the first chern classes mean that the determinant line bundles of $\mathbb{V}_\pm$ are diffeomorphically trivial since $c_1(\wedge^2\mathbb{V}_\pm) = c_1\mathbb{V}_\pm = 0$. Combining this with the isomorphisms (1) and (2) yields:

$$\wedge^2 p^*TM = \wedge^2(\mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+) = \mathcal{O}_Z(2) \otimes \wedge^2 p^*\mathbb{V}_+ = \mathcal{O}_Z(2) = \\ \mathcal{O}_Z(2) \otimes \wedge^2 p^*\mathbb{V}_- = \wedge^2(\mathcal{O}_Z(1) \otimes p^*\mathbb{V}_-) = \wedge^2(TF \oplus \mathcal{O}_Z) = \\ \oplus_{2=p+q}(\wedge^p TF \otimes \wedge^q \mathcal{O}_Z) = TF \otimes \mathcal{O}_Z = TF$$

since $TF$ is a line bundle. Taking the first chern class of both sides:

$$c_1(p^*TM) = c_1(\wedge^2 p^*TM) = c_1TF.$$

Alternatively, this chern class argument could be replaced with the previous taking wedge powers steps if the reader feels more comfortable with it.
Last equality implies the decomposition:

$$c_1Z = c_1(p^*TM \oplus TF) = c_1(p^*TM) + c_1TF = 2c_1TF.$$

So, $TF^*$ is a differentiable square root for the canonical bundle of $Z$. If $M$ is not spin $\mathbb{V}_\pm, \mathcal{O}_Z(1)$ are not globally defined, but the complex structure on their tensor product is still defined, and we can still use the isomorphisms (1),(2) for computing chern classes of the almost complex structure on $Z$ using differential forms defined locally by the metric. Consequently our decomposition is valid whether $M$ is spin or not.

One more word about the differentiable square roots is in order here. A differentiable square root implies a holomorphic one on complex manifolds since in the sheaf sequence:

$$\begin{array}{cccccccc} .. & \to & H^1(M,\mathcal{O}^*) & \to & H^2(M,\mathbb{Z}) & \to & .. \\ & & L & \mapsto & c_1(L) & \mapsto & 0 \\ & & & & \frac{1}{2}c_1(L) & \mapsto & 0 \end{array}$$

$c_1(L)$ maps to 0 since it is coming from a line bundle, and if it decomposes, $\frac{1}{2}c_1(L)$ maps onto 0 too, that means it is the first chern class of a line bundle.

# 5 Vanishing Theorem

Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-paramater standard deformation of $Z_0$, where $\mathcal{U} \subset \mathbb{C}$ is an open disk about the origin. Then the invertible sheaf $K_{\mathcal{Z}}$ has a square root as a holomorphic line bundle as follows:

We are going to show that the Steifel-Whitney class $w_2(K_{\mathcal{Z}})$ is going to vanish. We write $\mathcal{Z} = \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_i$ is a tubular neighborhood of $\widetilde{Z}_i$, $\mathcal{U}_1 \cap \mathcal{U}_2$ is a tubular neighborhood of $Q = \widetilde{Z}_1 \cap \widetilde{Z}_2$. So that $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2$ deformation retracts on $\widetilde{Z}_1, \widetilde{Z}_2$ and $Q$. Since $Q \approx \mathbb{P}_1 \times \mathbb{P}_1$ is simply connected, $H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) = 0$ and the map $r_{12}$ in the Mayer-Vietoris exact sequence :

$$\begin{array}{cccccccc} .. & \to & H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) & \to & H^2(\mathcal{U}_1 \cup \mathcal{U}_2, \mathbb{Z}_2) & \stackrel{r_{12}}{\to} & H^2(\mathcal{U}_1, \mathbb{Z}_2) \oplus H^2(\mathcal{U}_2, \mathbb{Z}_2) & \to & .. \\ & & \| & & \in & & & & \\ & & 0 & & w_2(K_{\mathcal{Z}}) & & & & \end{array}$$

is injective. Therefore it is enough to see that the restrictions $r_i(w_2(K_{\mathcal{Z}})) \in H^2(\mathcal{U}_i, \mathbb{Z}_2)$ are zero. For that, we need to see that $K_{\mathcal{Z}}|_{\widetilde{Z}_i}$ has a radical :

$$K_{\mathcal{Z}}|_{\widetilde{Z}_1} \stackrel{(1)}{=} (K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} \stackrel{(2)}{=} (K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1} \stackrel{(3)}{=} ((\pi^* K_{Z_1} + Q) + Q)|_{\widetilde{Z}_1} = 2(\pi^* K_{Z_1}^{1/2} + Q)|_{\widetilde{Z}_1}$$

where (1) is the application of the adjuction formula on $\widetilde{Z}_1$, $K_{\widetilde{Z}_1} = K_{\mathcal{Z}}|_{\widetilde{Z}_1} \otimes [\widetilde{Z}_1]$.

(2) comes from the linear equivalence of 0 with $Z_t$ on $\widetilde{Z}_1$, and $Z_t$ with $Z_0$:

$$0 = \mathcal{O}(Z_t)|_{\widetilde{Z}_1} = \mathcal{O}(Z_0)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + \widetilde{Z}_2)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + Q)|_{\widetilde{Z}_1}$$

(3) is the change of the canonical bundle under the blow up along a submanifold, see [GH]p608. $K_{Z_1}$ has a natural square root as we computed in the previous section, so $\pi^* K_{Z_1}^{1/2} \otimes [Q]$ is a square root of $K_{\mathcal{Z}}$ on $\widetilde{Z}_1$. Similarly on $\widetilde{Z}_2$, so $K_{\mathcal{Z}}$ has a square root $K_{\mathcal{Z}}^{1/2}$.

Before our vanishing theorem, we are going to mention the Semicontinuity Principle and the Hitchin's Vanishing theorem, which are involved in the proof:

**Lemma 5.1** (Semicontinuity Principle[Voisin])**.** *Let $\phi : \mathcal{X} \to \mathcal{B}$ be a family of complex compact manifolds With fiber $X_b, b \in \mathcal{B}$. Let $\mathcal{F}$ be a holomorphic vector bundle over $\mathcal{X}$, then*

*The function $b \mapsto h^q(X_b, \mathcal{F}|_{X_b})$ is upper semicontinuous. In other words, we have $h^q(X_b, \mathcal{F}|_{X_b}) \leq h^q(X_0, \mathcal{F}|_{X_0})$ for $b$ in a neighborhood of $0 \in \mathcal{B}$.*

**Lemma 5.2** (Hitchin Vanishing[HitLin][Poon86])**.** *Let $Z$ be the twistor space of an oriented self-dual riemannian manifold of positive scalar curvature with canonical bundle $K$, then*

$$h^0(Z, \mathcal{O}(K^{n/2})) = h^1(Z, \mathcal{O}(K^{n/2})) = 0 \text{ for all } n \geq 1.$$

**Theorem 5.3** (Vanishing Theorem)**.** *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-parameter standard deformation of $Z_0$, where $Z_0$ is as in Theorem (2.1), and $\mathcal{U} \subset \mathbb{C}$ is a neighborhood of the origin. Let $L \to \mathcal{Z}$ be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\widetilde{Z}_1}(K_{\mathcal{Z}}^{1/2})$$

*If $(M_i, [g_i])$ has positive scalar curvature, then by possibly replacing $\mathcal{U}$ with a smaller neighborhood of $0 \in \mathbb{C}$ and simultaneously replacing $\mathcal{Z}$ with its inverse image, we can arrange for our complex 4-fold $\mathcal{Z}$ to satisfy*

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

*Proof.* The proof proceeds by analogy to the techniques in [LeOM], and consists of several steps :

1. It is enough to **show that $\mathbf{H^j(Z_0, \mathcal{O}(L^*)) = 0}$ for $\mathbf{j = 1, 2}$ :** Since that would imply $h^j(Z_t, \mathcal{O}(L^*)) \leq 0$ for $j = 1, 2$ in a neighborhood by the semicontinuity principle. Intuitively, this means that the fibers are too small, so we can apply Proposition (3.3) to see $R^j\omega_*\mathcal{O}(L^*) = 0$ for $j = 1, 2$. The first page of the Leray Spectral Sequence reads :

$$
E_2 \quad \begin{array}{|cccc}
\vdots & \vdots & \vdots & \\
H^0(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & \cdots \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
H^0(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & \cdots \\
\hline
\end{array}
$$

Remember that

$$E_2^{p,q} = H^p(\mathcal{U}, R^q\omega_*\mathcal{O}(L^*))$$

$$E_\infty^{p,q} = H^{p+q}(\mathcal{Z}, \mathcal{O}(L^*))$$

and that the differential

$$d_2(E_2^{p,q}) \subset E_2^{p+2,q-1}.$$

Vanishing of the second row implies the immediate convergence of the first row till the third column because of the differentials, so

$$E_\infty^{p,0} = E_2^{p,0} \text{ i.e. } H^{p+0}(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) \text{ for } p \leq 3$$

hence $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*))$ , for $p \leq 3$.
Since $\mathcal{U}$ is one dimensional, $\omega : \mathcal{Z} \to \mathcal{U}$ has to be a flat morphism, so the sheaf $\omega_*\mathcal{O}(L^*)$ is coherent[Gun, Bon]. $\mathcal{U}$ is an open subset of $\mathbb{C}$ implying that it is Stein. And the so called Theorem B of Stein Manifold theory characterizes them as possesing a vanishing higher dimensional($p > 0$) coherent sheaf cohomology [Lew]p67, [H]p252, [Gun, Bon]. So $H^p(\mathcal{U}, \omega_*\mathcal{O}(L^*)) = 0$ for $p > 0$. Tells us that $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ for $0 < p \leq 3$.

2. Related to $Z_0$, we have the **Mayer-Vietoris like** sheaf exact sequence

$$0 \to \mathcal{O}_{Z_0}(L^*) \to \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*) \to \mathcal{O}_Q(L^*) \to 0$$

where $\nu : \widetilde{Z}_1 \sqcup \widetilde{Z}_2 \to Z_0$ is the inclusion map on each of the two components of the disjoint union $\widetilde{Z}_1 \sqcup \widetilde{Z}_2$. This gives the long exact cohomology sequence piece :

$$\begin{gathered} 0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to \\ H^1(\mathcal{O}_Q(L^*)) \to H^2(\mathcal{O}_{Z_0}(L^*)) \to \\ H^2(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to 0 \end{gathered}$$

due to the fact that :

3. $\mathbf{H^0}(\mathcal{O}_\mathbf{Q}(\mathbf{L}^*)) = \mathbf{H^2}(\mathcal{O}_\mathbf{Q}(\mathbf{L}^*)) = \mathbf{0}$ **:** To see this, we have to understand the restriction of $\mathcal{O}(L^*)$ to $Q$ :

$$\begin{gathered} L^*|_Q = (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}(K_{\widetilde{Z}_2} - \widetilde{Z}_2) - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = \\ (\tfrac{1}{2}(K_{\widetilde{Z}_2} + Q) - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}(K_{\widetilde{Z}_2} - Q)|_{\widetilde{Z}_2}|_Q = \\ \tfrac{1}{2}(K_Q - Q - Q)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}K_Q - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}K_Q|_Q \otimes NQ_{\widetilde{Z}_2}^{-1} = \\ \mathcal{O}(-2,-2)^{1/2} \otimes \mathcal{O}(1,-1)^{-1} = \mathcal{O}(-2,0) \end{gathered}$$

here, we have computed the normal bundle of $Q$ in $\widetilde{Z}_2$ in Lemma (2.2) as $\mathcal{O}(1,-1)$, where the second component is the fiber direction in the blowing up process. So the line bundle $L^*$ is trivial on the fibers. Since $Q = \mathbb{P}_1 \times \mathbb{P}_1$, we have

$$\begin{gathered} H^0(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^0(\mathbb{P}_1 \times \mathbb{P}_1, \pi_1^*\mathcal{O}(-2)) = \\ H^0(\mathbb{P}_1, \pi_{1*}\pi_1^*\mathcal{O}(-2)) = H^0(\mathbb{P}_1, \mathcal{O}(-2)) = 0 \end{gathered}$$

by the Leray spectral sequence and the projection formula since $H^k(\mathbb{P}_1, \mathcal{O}) = 0$ for $k > 0$. Similarly

$$H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^2(\mathbb{P}_1, \mathcal{O}(-2)) = 0$$

by dimensional reasons. Moreover, for the sake of curiosity

$$\begin{gathered} H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^1(\mathbb{P}_1, \mathcal{O}(-2)) \approx \\ H^0(\mathbb{P}_1, \mathcal{O}(-2) \otimes \mathcal{O}(-2)^*)^* = H^0(\mathbb{P}_1, \mathcal{O})^* = \mathbb{C}. \end{gathered}$$

4. $\mathbf{H^1(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = H^2(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = 0}$ **:** These are applications of Hitchin's second Vanishing Theorem and are going to help us to simplify our exact sequence piece.

$$H^1(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - \widetilde{Z}_1)|_{\widetilde{Z}_2}) = \\ H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - Q)|_{\widetilde{Z}_2}) = H^1(\widetilde{Z}_2, \pi^* K_{Z_2}^{1/2}) = \\ H^1(Z_2, \pi_*\pi^* K_{Z_2}^{1/2}) = H^1(Z_2, K_{Z_2}^{1/2}) = 0$$

by the Leray spectral sequence, projection formula and Hitchin's Vanishing theorem for $Z_2$, since it is the twistor space of a positive scalar curvature space. This implies $H^2(Z_2, K_{Z_2}^{1/2}) \approx H^1(Z_2, K_{Z_2}^{1/2})^* = 0$ because of the Kodaira-Serre Duality. Hence our cohomological exact sequence piece simplifies to

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to H^1(\mathcal{O}_Q(L^*)) \to \\ H^2(\mathcal{O}_{Z_0}(L^*)) \to H^2(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to 0$$

5. $\mathbf{H^k(\mathcal{O}_{\widetilde{Z}_1}(L^* \otimes [Q]_{\widetilde{Z}_1}^{-1})) = 0}$ **for** $\mathbf{k = 1, 2, 3}$ **:** This technical result is going to be needed to understand the exact sequence in the next step. First we simplify the sheaf as

$$(L^* - Q)|_{\widetilde{Z}_1} \overset{def}{=} (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1 - Q)|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}}|_{\widetilde{Z}_1} \overset{adj}{=} \\ \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} = \tfrac{1}{2}(K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1}.$$

So

$$H^k(\widetilde{Z}_1, L^* - Q) = H^k(\widetilde{Z}_1, (K_{\widetilde{Z}_1} + Q)/2) \overset{sd}{\approx} H^{3-k}(\widetilde{Z}_1, (K_{\widetilde{Z}_1} - Q)/2)^* \\ = H^{3-k}(\widetilde{Z}_1, \tfrac{1}{2}\pi^* K_{Z_1})^* \overset{lss}{=} H^{3-k}(Z_1, \tfrac{1}{2}\pi_*\pi^* K_{Z_1})^* \\ \overset{pf}{=} H^{3-k}(Z_1, K_{Z_1}^{1/2})^* \overset{sd}{\approx} H^k(Z_1, K_{Z_1}^{1/2})$$

and one of the last two terms vanish in any case for $k = 1, 2, 3$. So we apply the Hitchin Vanishing theorem for dimensions 0 and 1.

6. **Restriction maps to Q :** Consider the exact sequence of sheaves on $\widetilde{Z}_1$ :

$$0 \to \mathcal{O}_{\widetilde{Z}_1}(L^* \otimes [Q]_{\widetilde{Z}_1}^{-1}) \to \mathcal{O}_{\widetilde{Z}_1}(L^*) \to \mathcal{O}_Q(L^*) \to 0.$$

The previous step implies that the restriction maps :

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_1} H^1(\mathcal{O}_Q(L^*))$$

and

$$H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_2} H^2(\mathcal{O}_Q(L^*))$$

are isomorphism. In particular $H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) = 0$ due to (3). Incidentally, this exact sheaf sequence is a substitute for the role played by the Hitchin Vanishing Theorem, for the $\widetilde{Z}_2$ components in the cohomology sequence. It also assumes Hitchin's theorems for the $\widetilde{Z}_1$ component.

7. **Conclusion :** Our cohomology exact sequence piece reduces to

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_1} H^1(\mathcal{O}_Q(L^*)) \to H^2(\mathcal{O}_{Z_0}(L^*)) \to 0$$

the isomorphism in the middle forces the rest of the maps to be 0 and hence we get $H^1(\mathcal{O}_{Z_0}(L^*)) = H^2(\mathcal{O}_{Z_0}(L^*)) = 0$.

$\square$

## The Sign of the Scalar Curvature

The sections after this point are devoted to detect the sign of the scalar curvature of the metric we consider on the connected sum. We use Green's Functions for that purpose. Positivity for the scalar curvature is going to be characterized by non-triviality of the Green's Functions. Then our Vanishing Theorem will provide the Serre-Horrocks vector bundle construction, which gives the Serre Class, a substitute for the Green's Function by Atiyah[AtGr]. And non-triviality of the Serre Class will provide the non-triviality of the extension described by it.

# 6 Green's Function Characterization

In this section, we define the Green's Functions. To get a unique Green's Function, we need an operator which has a trivial kernel. So we begin with a compact Riemannian 4-manifold $(M, g)$, and assume that its *Yamabe Laplacian* $\Delta + s/6$ has trivial kernel.This is automatic if $g$ is conformally equivalent to a metric of positive scalar curvature, impossible if it is conformally equivalent to a metric of zero scalar curvature because of the Hodge Laplacian, and may or may not happen for a metric of negative scalar curvature. Since the Hodge Laplacian $\Delta$ is self-adjoint, $\Delta + s/6$ is also self-adjoint implying that it has a trivial cokernel, if once have a trivial kernel. Therefore it is a bijection and we have a unique smooth solution $u$ for the equation $(\Delta + s/6)u = f$ for any smooth function $f$. It also follows that it has a unique distributional solution $u$ for any distribution $f$. Let $y \in M$ be any point. Consider the Dirac delta distribution $\delta_y$ at $y$ defined by

$$\delta_y : C^\infty(M) \to \mathbb{R} \quad , \quad \delta_y(f) = f(y)$$

intuitively, this behaves like a function identically zero on $M - \{y\}$, and infinity at $y$ with integral 1. Then there is a unique distributional solution $G_y$ to the equation

$$(\Delta + s/6)G_y = \delta_y$$

called the *Green's Function* for $y$. Since $\delta_y$ is identically zero on $M - \{y\}$, elliptic regularity implies that $G_y$ is smooth on $M - \{y\}$.

About $y$, one has an expansion

$$G_y = \frac{1}{4\pi^2}\frac{1}{\varrho_y^2} + O(\log \varrho_y)$$

near $\varrho_y$ denotes the distance from $y$. In the case $(M, g)$ is self-dual this expansion reduces to [AtGr]

$$G_y = \frac{1}{4\pi^2}\frac{1}{\varrho_y^2} + \textit{bounded terms}$$

We also call $G_y$ to be the *conformal Green's function* of $(M, g, y)$.

This terminology comes from the fact that the Yamabe Laplacian is a *conformally invariant* differential operator as a map between sections of some real line bundles. For any nonvanishing smooth function $u$, the conformally equivalent metric $\tilde{g} = u^2 g$ has scalar curvature

$$\tilde{s} = 6u^{-3}(\Delta + s/6)u$$

A consequence of this is that $u^{-1}G_y$ is the conformal Green's function for $(M, u^2 g, y)$ if $G_y$ is the one for $(M, g, y)$.

Any metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature sign. Since if $u \not\equiv 0$ is the eigenfunction of the lowest eigenvalue $\lambda$ of the Yamabe Laplacian,

$$\tilde{s} = 6u^{-3}\lambda u = 6\lambda u^{-2}$$

for the metric $\tilde{g} = u^2 g$. Actually a more stronger statement is true thanks to the proof[LP] of the Yamabe Conjecture, any metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature(CSC). Also if two metrics with scalar curvatures of fixed signs are conformally equivalent, then their scalar curvatures have the same sign.

The sign of Yamabe constant of a conformal class, meaning the sign of the constant scalar curvature of the metric produced by the proof of the Yamabe conjecture is the same as the sign of the smallest Yamabe eigenvalue $\lambda$ for any metric in the conformal class.

Before giving our characterization for positivity, we are going to state the maximum principle we will be using. Consider the differential operator $L_c = \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$ arranged so that $a^{ij} = a^{ji}$. It is called *elliptic [PrWe]* at a point $x = (x_1..x_n)$ if there is a positive quantity $\mu(x)$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \mu(x)\sum_{i=1}^{n} {\xi_i}^2$$

for all $n$-tuples of real numbers $(\xi_1..\xi_n)$. The operator is said to be uniformly elliptic in a domain $\Omega$ if the inequality holds for each point of $\Omega$ and if there is a positive constant $\mu_0$ such that $\mu(x) \geq \mu_0$ for all $x$ in $\Omega$. Ellipticity of a more general second order operator is defined via its second order term.

In the matrix language, the ellipticity condition asserts that the symmetric matrix $[a^{ij}]$ is positive definite at each point $x$.

**Lemma 6.1** (Hopf's strong maximum principle [PrWe]). *Let $u$ satisfy the differential inequality*

$$(L_c + h)u \geq 0 \quad with \quad h \leq 0$$

*where $L_c$ is uniformly elliptic in $\Omega$ and coefficients of $L_c$ and $h$ bounded. If $u$ attains a nonnegative maximum at an interior point of $\Omega$, then $u$ is constant.*

So if for example the maximum of $u$ is attained in the interior and is 0, then $u$ has to vanish. An application of this principle provides us with a criterion of determining the sign of the Yamabe Constant using Green's Functions:

**Lemma 6.2** (Green's Function Characterization for the Sign[LeOM])**.** *Let $(M, g)$ be a compact Riemannian 4-manifold with $Ker(\Delta + s/6) = 0$, i.e. the Yamabe Laplacian has trivial kernel, taking $\Delta = d^*d$[AtGr]. Fix a point $y \in M$. Then for the conformal class $[g]$ we have the following assertions :*
*1. It does not contain a metric of zero scalar curvature*
*2. It contains a metric of positive scalar curvature iff $G_y(x) \neq 0$ for all $x \in M - \{y\}$*
*3. It contains a metric of negative scalar curvature iff $G_y(x) < 0$ for some $x \in M - \{y\}$*

*Proof.* Proceeding as in [LeOM], $[g]$ has three possibilities for its Yamabe Type, one of 0,+,−. Since the Yamabe Laplacian is conformally invariant as acting on functions with conformal weight, we assume that either $s = 0$ or $s > 0$ or else $s < 0$ everywhere.

$s = 0$ : Then $(\Delta + 0/6)f = \Delta f = 0$ is solved by any nonzero constant function $f$. Therefore $Ker(\Delta + s/6) \neq 0$, which is not our situation.

$s > 0$ : For the smooth function $G_y : M - \{y\} \to \mathbb{R}$ , $G_y^{-1}((-\infty, a])$ is closed hence compact for any $a \in \mathbb{R}$. Hence it has a minimum say at $m$ on $M - \{y\}$. We also have $(\Delta + s/6)G_y = 0$ on $M - \{y\}$. At the minimum, choose normal coordinates so that $\Delta G_y(m) = -\sum_{k=1}^{4} \partial_k^2 G_y(m)$. Second partial derivatives are greater than or equal to zero, $\Delta G_y(m) \leq 0$ so $G_y(m) = -\frac{6}{s}\Delta G_y(m) \geq 0$. We got nonnegativity, but need positivity, so assume $G_y(m) = 0$.
Then the maximum of $-G_y$ is attained and it is nonnegative with $(\Delta_c - s/6)(-G_y) = 0 \geq 0$. So the strong maximum principle(6.1) is applicable and $-G_y \equiv 0$. This is impossible since $G_y(x) \to \infty$ as $x \to y$, hence $m \neq 0$ and $G_y > 0$. Note that the weak maximum principle was not applicable since we had $G_y \geq 0$, implied $\Delta_c G_y = \frac{s}{6} G_y \geq 0$ though we got a minimum rather than a maximum. Also note that $\nabla G_y(m) = 0$ at a minimum though this does not imply $div \nabla G_y(m) = 0$.

$s < 0$ : In this situation we have

$$\frac{1}{6}\int_M sG_y dV = \int_M (\Delta + s/6)G_y dV = \int_M \delta_y dV = 1 > 0$$

implying $G_y < 0$ at some point. Besides, at some other point it should be zero since $G_y(x) \to +\infty$ as $x \to y$.

$\square$

# 7 Cohomological Characterization

Now let $(M^4, g)$ be a compact self-dual Riemannian manifold with the twistor space $Z$. One of the basic facts of the twistor theory[HitLin] is that for any open set $U \subset M$ and the correponding inverse image $\widetilde{U} \subset Z$ in the twistor space, there is a natural isomorphism

$$pen : H^1(\widetilde{U}, \mathcal{O}(K^{1/2})) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{smooth complex-valued solutions} \\ \text{of } \ (\Delta + s/6)u = 0 \ \text{ in } \ U \end{array} \right\}$$

which is called *the Penrose transform*[BaSi, HitKä, AtGr], where $K = K_Z$. Since locally $\mathcal{O}(K^{1/2}) \approx \mathcal{O}(-2)$ e.g. $Z = \mathbb{CP}_3$, for a cohomology class $\psi \in H^1(\widetilde{U}, \mathcal{O}(K^{1/2}))$, the value of the corresponding function $pen_\psi$ at $x \in U$ is obtained by restricting $\psi$ to the twistor line $P_x \subset Z$ to obtain an element

$$pen_\psi(x) = \psi|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2})) \approx H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \approx \mathbb{C}.$$

Note that $pen_\psi$ is a section of a line bundle, but the choice of a metric $g$ in the conformal class determines a canonical trivialization of this line bundle [HitKä], and $pen_\psi$ then becomes an ordinary function. Taking $U = M - \{y\}$ we have $(\Delta + s/6)G_y = 0$ on $U$ in the uniquely presence of the conformal Green's functions(6) and $G_y(x)$ is regarded as a function of $x$ corresponds to a canonical element

$$pen^{-1}(G_y) \in H^1(Z - P_y, \mathcal{O}(K^{1/2}))$$

where $P_y$ is the twistor line over the point $y$.

What is this interesting cohomology class? The answer was discovered by Atiyah [AtGr] involving the *Serre Class* of a complex submanifold. Which is a construction due to Serre [Ser] and Horrocks [Hor]. We now give the definition of the Serre class via the following lemma:

**Lemma 7.1** (Serre-Horrocks Vector Bundle,Serre Class)**.** *Let $W$ be a (possibly non-compact) complex manifold, and let $V \subset W$ be a closed complex submanifold of complex codimension 2, and $N = N_{V/W}$ be the normal bundle of $V$. For any holomorphic line bundle $L \to W$ satisfying*

$$L|_V \approx \wedge^2 N \quad \text{and} \quad H^1(W, \mathcal{O}(L^*)) = H^2(W, \mathcal{O}(L^*)) = 0$$

*There is a rank-2 holomorphic vector bundle $E \to W$ called the* Serre-Horrocks bundle *of $(W, V, L)$ , together with a holomorphic section $\zeta$ satisfying*

$$\wedge^2 E \approx L \quad , \quad d\zeta|_V : N \xrightarrow{\sim} E \quad \text{and} \quad \zeta = 0 \text{ exactly on V.}$$

*The pair $(E, \zeta)$ is unique up to isomorphism if we also impose that the isomorphism $\det d\zeta : \wedge^2 N \to \wedge^2 E|_V$ should agree with a given isomorphism $\wedge^2 N \to L|_V$. They also give rise to an extension*

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_V \to 0,$$

*the class of which is defined to be the* Serre Class $\lambda(V) \in \mathrm{Ext}^1_W(\mathscr{I}_V, \mathcal{O}(L^*))$, *where $\mathscr{I}_V$ is the ideal sheaf of $V$, and this extension determines an element of $H^1(W - V, \mathcal{O}(L^*))$ by restricting to $W - V$.*

*Proof.* Consult [LeOM] for a proof. □

For an alternative treatment of Serre's class via the Grothendieck class consult [AtGr]. We are now ready to state the answer of Atiyah :

**Theorem 7.2** (Atiyah[AtGr])**.** *Let $(M^4, g)$ be a compact self-dual Riemannian manifold with twistor space $Z$, and assume that the conformally invariant Laplace operator $\Delta = d^*d + s/6$ on $M$ has no global nontrivial solution so that the Green's functions are well defined. Let $y \in M$ be any point, and $P_y \subset Z$ be the corresponding twistor line.*
*Then the image of the Serre class $\lambda(P_y) \in \mathrm{Ext}^1_Z(\mathscr{I}_{P_y}, \mathcal{O}(K^{1/2}))$ in $H^1(Z - P_y, \mathcal{O}(K^{1/2}))$ is the Penrose transform of the Green's function $G_y$ times a non-zero constant. More precisely*

$$pen^{-1}(G_y) = \frac{1}{4\pi^2}\lambda(P_y)$$

Now thanks to this remarkable result of Atiyah, we can substitute the Serre class for the Green's functions in our previous characterization 6.2 and get rid of them to obtain a better criterion for positivity as follows :

**Proposition 7.3** (Cohomological Characterization , [LeOM]). *Let $(M^4, g)$ be a compact self-dual Riemannian manifold with twistor space $Z$. Let $P_y$ be a twistor line in $Z$.*
*Then the conformal class $[g]$ contains a metric of positive scalar curvature if and only if $H^1(Z, \mathcal{O}(K^{1/2})) = 0$, and the Serre-Horrocks vector bundle(7.1) on $Z$ taking $L = K^{-1/2}$ associated to $P_y$ satisfies $E|_{P_x} \approx \mathcal{O}(1) \oplus \mathcal{O}(1)$ for every twistor line $P_x$*

*Proof.* $\Rightarrow$ : If a conformal class contains a metric of positive scalar curvature $g$, then we can show that $Ker(\Delta + \frac{s}{6})$ is trivial as follows: Let $(\Delta + \frac{s}{6})u = 0$ for some smooth function $u : M \to \mathbb{R}$ and $s > 0$. Since $M$ is compact, $u$ has a minimum say at some point $m$. At the minimum one has

$$\Delta u(m) = -\sum u_{kk}(m) \leq 0$$

because of the normal coordinates about $m$, modern Laplacian and second derivative test. So that

$$\Delta u = -\frac{su}{6} \leq 0 \quad \text{implying} \quad u \geq 0 \quad \text{everywhere.}$$

If we integrate over $M$ on gets 0 for the Laplacian of a function so

$$0 = \int_M \Delta u \; dV = \int_M -\frac{su}{6} dV$$

hence

$$\int_M su \; dV = 0 \quad \text{implying} \quad u \equiv 0 \quad \text{since} \quad s > 0$$

that is to say that the kernel is zero.

Remember the Penrose Transform map

$$pen : H^1(M, \mathcal{O}(K^{1/2})) \xrightarrow{\sim} Ker(\Delta + \frac{s}{6})$$

implies that $H^1(M, \mathcal{O}(K^{1/2})) = 0$, also by Serre Duality

$$H^2(M, K^{1/2}) \approx H^{0,2}_{\bar{\partial}}(M, K^{1/2}) \overset{SD}{\approx} H^{3,1}_{\bar{\partial}}(M, K^{1/2*})^* \approx H^1(M, K \otimes K^{-1/2})^*$$

$$= H^1(M, K^{1/2})^* = 0$$

also

$$\wedge^2 NP_y = \wedge^2 \mathcal{O}_{P_y}(1) \oplus \mathcal{O}_{P_y}(1) = \bigoplus_{2=p+q} \wedge^p \mathcal{O}(1) \otimes \wedge^q \mathcal{O}(1) = \wedge^1 \mathcal{O}(1) \otimes \wedge^1 \mathcal{O}(1)$$

$$= \mathcal{O}_{\mathbb{P}_1}(2) = K^{-1/2}|_{P_y}$$

since $K^{-1/2}|_{P_y} = \mathcal{O}_{\mathbb{P}_3}(4)^{1/2}|_{P_y} = \mathcal{O}_{P_y}(2)$. So that the hypothesis for the Serre-Horrocks vector bundle construction (7.1) for $L = K^{-1/2}$ is satisfied. Then we have the image of the Serre class

$$4\pi^2 pen^{-1}(G_y) = \lambda(P_y) \in H^1(Z - P_y, K^{1/2})$$

So

$$4\pi^2 G_y(x) = pen_{\lambda(P_y)}(x) = \lambda(P_y)|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2})) \approx \mathbb{C}$$

where

$$H^1(P_x, \mathcal{O}(K^{1/2})) \approx H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \approx H^0(\mathbb{CP}_1, \Omega^1(\mathcal{O}(-2)^*)) = H^0(\mathbb{CP}_1, \mathcal{O}) \approx \mathbb{C}$$

By the Green's Function Characterization (6.2) we know that $4\pi^2 G_y(x) \neq 0$. So $\lambda(P_y)|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2}))$ is also nonzero.

Since $\lambda(P_y)$ corresponds to the extension

$$0 \to \mathcal{O}(K^{1/2}) \to \mathcal{O}(E^*) \to \mathcal{I}_{P_y} \to 0$$

If we restrict to $Z - P_y$

$$0 \to \mathcal{O}(K^{1/2}) \to \mathcal{O}(E^*) \to \mathcal{O} \to 0$$

dualizing we obtain

$$0 \to \mathcal{O} \to \mathcal{O}(E) \to \mathcal{O}(K^{-1/2}) \to 0$$

now restricting this extension to $P_x$

$$0 \to \mathcal{O}_{\mathbb{P}_1} \to \mathcal{O}(E)|_{P_x} \to \mathcal{O}(2) \to 0$$

So since $G_y(x) \neq 0$ , we expect that this extension is nontrivial. Let's figure out the possibilities. First of all, by the theorem of Grothendieck [VB]p22 every holomorphic vector bundle over $\mathbb{P}_1$ splits. In our case $E|_{P_x} = \mathcal{O}(k) \oplus \mathcal{O}(l)$ for some $k, l \in \mathbb{Z}$. Moreover if we impose $k \geq l$, this splitting is uniquely determined[VB].

Secondly, any short exact sequence of vector bundles splits topologically by [VB]p16. In our case, topologically we have $E|_{P_x} \stackrel{t}{=} \mathcal{O} \oplus \mathcal{O}(2)$. So, setting the chern classes to each other we have

$$c_1(E|_{P_x})[P_x] = c_1(\mathcal{O}(k) \oplus \mathcal{O}(l))[\mathbb{P}_1] = c_1\mathcal{O}(k) + c_1\mathcal{O}(l)[\mathbb{P}_1] = k + l$$

equal to

$$c_1(E|_{P_x})[P_x] = c_1(\mathcal{O} \oplus \mathcal{O}(2))[\mathbb{P}_1] = c_1\mathcal{O} + c_1\mathcal{O}(2)[\mathbb{P}_1] = 0 + 2 = 2.$$

Hence $l = 2 - k$. We now have $E|_{P_x} = \mathcal{O}(k) \oplus \mathcal{O}(2-k)$. Our extension becomes

$$0 \to \mathcal{O}_{\mathbb{P}_1} \to \mathcal{O}(k) \oplus \mathcal{O}(2-k) \to \mathcal{O}(2) \to 0$$

The inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(k) \oplus \mathcal{O}(2-k)$ gives a trivial holomorphic subbundle. It has one complex dimensional space of sections. So these sections are automatically sections of $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$, too. This implies

$$0 \neq H^0(\mathcal{O}(k) \oplus \mathcal{O}(2-k)) = H^0(\mathcal{O}(k)) \oplus H^0(\mathcal{O}(2-k))$$

Imposing $k, 2-k \geq 0$ by the Kodaira Vanishing Theorem[GH] since the direct sum elements $\mathcal{O}(k)$ and $\mathcal{O}(2-k)$ should possess sections. Also, from uniqueness $k \geq l = 2-k$. Altogether we have $2 \geq k \geq 1$. From the two choices, $k = 2$ gives the trivial extension $\mathcal{O}(2) \oplus \mathcal{O}$, $k = 1$ gives the nontrivial extension $E|_{P_x} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ as we expected. See the following remark for existence.

$\Leftarrow$ : For the converse, if $E|_{P_x} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ then we already showed that this is the nontrivial extension hence $G_y(x) \neq 0$, so that the scalar curvature is positive by the Green's Function Characterization (6.2) $\square$

**Remark 7.4.** *The nontrivial extension of $\mathcal{O}$ by $\mathcal{O}(2)$ exists by the Euler exact sequence*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \xrightarrow{\mathcal{E}} T'\mathbb{P}^n \to 0$$

*[GH]p409 for $n = 1$. Alternatively, the maps $i : \rho \mapsto (\rho Z_0, \rho Z_1)$ and $j : (u, v) \mapsto uZ_1 - vZ_0$ for coordinates $[Z_0 : Z_1]$ on $\mathbb{P}_1$ yields the exact sheaf sequence*

$$0 \to \mathcal{O}(-1) \xrightarrow{i} \mathcal{O} \oplus \mathcal{O} \xrightarrow{j} \mathcal{O}(1) \to 0$$

*tensoring with $\mathcal{O}(1)$ produces the nontrivial $\mathcal{O}(1) \oplus \mathcal{O}(1)$ extension. Since we have a unique nontrivial extension, this shows*

$$\mathrm{Ext}^1(\mathcal{O}(2), \mathcal{O}) = \mathbb{C}$$

*used in [AtGr] to classify the extensions. On the other hand*

$$H^1(Hom(\mathcal{O}(2), \mathcal{O})) = H^1(\mathcal{O}(2)^* \otimes \mathcal{O}) = H^1(\mathcal{O}_{\mathbb{P}_1}(-2))$$

$$= H^0(\mathcal{O}(2) \otimes \mathcal{O}(-2)^*) = H^0(\mathbb{P}_1, \mathcal{O}) = \mathbb{C}$$

*used in [DF] to classify the extensions. So, our computation verifies the isomorphism*

$$\mathrm{Ext}^q(M, \mathcal{F}, \mathcal{G}) \approx H^q(M, \mathcal{F}^* \otimes_{\mathcal{O}} \mathcal{G})$$

*for locally free sheaves or vector bundles for $q = 1$. See [GH]p706. Here,* Ext *stands for what is called the* global Ext *group usually defined to be the hypercohomology of the complex of sheaves associated to a global syzygy for $\mathcal{F}$. Though practically usually computed via the spectral sequence to be*

$$\mathrm{Ext}^k(\mathcal{F}, \mathcal{G}) = H^0(\underline{Ext}^k_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$$

*under some vanishing conditions[GH].* □

# 8 The Sign of the Scalar Curvature

We are now ready to approach the problem of determining the sign of the Yamabe constant for the self-dual conformal classes constructed in Theorem (2.1). The techniques used here are analogous to the ones used by LeBrun in [LeOM].

**Theorem 8.1.** *Let $(M_1, g_1)$ and $(M_2, g_2)$be compact self-dual Riemannian 4-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature.*

*Then, for all sufficiently small $\mathfrak{t} > 0$, the self-dual conformal class $[g_{\mathfrak{t}}]$ obtained on $M_1 \# M_2$ by the Donaldson-Friedman Theorem (2.1) contains a metric of positive scalar curvature.*

*Proof.* Pick a point $y \in (M_1 \# M_2) \backslash M_1$. Consider the real twistor line $P_y \subset \widetilde{Z}_2$, and extend this as a 1-parameter family of twistor lines in $P_{y_{\mathfrak{t}}} \subset Z_{\mathfrak{t}}$ for $\mathfrak{t}$ near $0 \in \mathbb{C}$ and such that $P_{y_{\mathfrak{t}}}$ is a real twistor line for $\mathfrak{t}$ real. By shrinking $\mathcal{U}$ if needed, we may arrange that $\mathcal{P} = \cup_{\mathfrak{t}} P_{y_{\mathfrak{t}}}$ is a closed codimension-2 submanifold of $\mathcal{Z}$ and $H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ by the Vanishing Theorem (5.3). Next we check that $L|_{\mathcal{P}} \approx \wedge^2 N_{\mathcal{P}}$. Over a twistor line $P_{y_{\mathfrak{t}}}$ we have

$$\wedge^2 N_{\mathcal{P}}|_{P_{y_{\mathfrak{t}}}} = \wedge^2(\mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathcal{O}_{P_{y_{\mathfrak{t}}}}(2)$$

by considering the first Chern classes. On the other hand, notice that the restriction of $L^*$ to any smooth fiber $Z_{\mathfrak{t}}$, $\mathfrak{t} \neq 0$ is simply $K^{1/2}$ :

$$L^*|_{Z_{\mathfrak{t}}} = (\frac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1)|_{Z_{\mathfrak{t}}} = \frac{1}{2}K_{\mathcal{Z}}|_{Z_{\mathfrak{t}}} = \frac{1}{2}(K_{Z_{\mathfrak{t}}} - Z_{\mathfrak{t}})|_{Z_{\mathfrak{t}}} = \frac{1}{2}K_{Z_{\mathfrak{t}}}|_{Z_{\mathfrak{t}}}.$$

Here, $\widetilde{Z}_1|_{Z_{\mathfrak{t}}} = 0$ because of the fact that $\widetilde{Z}_1$ and $Z_{\mathfrak{t}}$ does not intersect for $\mathfrak{t} \neq 0$. The normal bundle of $Z_{\mathfrak{t}}$ is trivial, because of the fact that we have a standard deformation. Then

$$L|_{P_{y_{\mathfrak{t}}}} = K_{Z_{\mathfrak{t}}}^{-1/2}|_{P_{y_{\mathfrak{t}}}} = TF|_{P_{y_{\mathfrak{t}}}} = \mathcal{O}_{P_{y_{\mathfrak{t}}}}(2) \quad \text{for} \quad \mathfrak{t} \neq 0$$

since $TF$ of Sec (4) is the square-root of the anti-canonical bundle. For the case $\mathfrak{t} = 0$, we need the fact that $L^*|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{1/2}$ which we have computed in the step 4 of the proof of the vanishing theorem (5.3). This yields

$$L|_{P_{y_0}} = \pi^* K_{Z_2}^{-1/2}|_{\widetilde{Z}_2}|_{P_{y_0}} = \mathcal{O}_{P_{y_0}}(2).$$

Then the Serre-Horrocks construction (7.1) is available to obtain the holomorphic vector bundle $E \to \mathcal{Z}$ and a holomorphic section $\zeta$ vanishing exactly along $\mathcal{P}$, also, the corresponding extension

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_{\mathcal{P}} \to 0$$

gives us the Serre class $\lambda(\mathcal{P}) \in H^1(\mathcal{Z} - \mathcal{P}, \mathcal{O}(L^*))$.

Since $L^*|_{Z_{\mathfrak{t}}} = K_{Z_{\mathfrak{t}}}^{1/2}$ for $\mathfrak{t} \neq 0$ by the above computation, Proposition (7.2) of Atiyah tells us that the restriction of $\lambda(\mathcal{P})$ to $Z_{\mathfrak{t}}$, $\mathfrak{t} > 0$, has Penrose transform equal to a positive constant times the conformal Green's function of $(M_1 \# M_2, g_{\mathfrak{t}}, y_{\mathfrak{t}})$ for any $\mathfrak{t} > 0$.

Now, we will restrict $(E, \zeta)$ to the two components of the divisor $Z_0$. We begin by restricting to $\widetilde{Z}_2$. We have $L|_{P_{y_0}} = \mathcal{O}_{P_0}(2) = \wedge^2 N P_{y_0}$ and

$$H^k(\widetilde{Z}_2, L^*) = H^k(\widetilde{Z}_2, \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, \pi_* \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, K_{Z_2}^{1/2}) = 0$$

for $k = 1, 2$ because of the projection lemma, Leray spectral sequence and the Hitchin's Vanishing theorem for positive scalar curvature on $M_2$. So that we have the Serre-Horrocks bundle for the triple $(\widetilde{Z}_2, P_{y_0}, L|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{-1/2})$. On the other hand it is possible to construct the Serre-Horrocks bundle $E_2$ for the triple $(Z_2, P_{y_0}, K_{Z_2}^{-1/2})$ for which all conditions are already checked to be satisfied. In the construction of these Serre-Horrocks bundles, if we stick to a chosen isomorphism $\wedge^2 N \to L|_{P_{y_0}}$, these bundles are going to be isomorphic by (7.1). The splitting type of $E$ on the twistor lines corresponding to the points in $M_2 - \{y_0, p_2\}$ supposed to be the same as the splitting type of $E_2$, which is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ since $Z_2$ already admits a self-dual metric of positive scalar curvature.

Secondly, we restrict $(E, \zeta)$ to $\widetilde{Z}_1$. Alternatively we restrict the Serre class $\lambda(\mathcal{P})$ to $H^1(\widetilde{Z}_1, \mathcal{O}(L^*))$ where

$$L^*|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}} + Q|_{\widetilde{Z}_1} \overset{adj}{=} \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1) + Q|_{\widetilde{Z}_1} =$$
$$\tfrac{1}{2}(K_{\widetilde{Z}_1} + Q) + Q|_{\widetilde{Z}_1} = \tfrac{1}{2}(\pi^* K_{Z_1} + 2Q) + Q|_{\widetilde{Z}_1} = \pi^* \tfrac{1}{2} K_{Z_1} + 2Q|_{\widetilde{Z}_1},$$

and show that it is non-zero on every real twistor line away from $Q$ here. Remember that we have the the restriction isomorphism obtained in the step 6 of the proof of the vanishing theorem (5.3)

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\sim} H^1(\mathcal{O}_Q(L^*)) \approx \mathbb{C}$$

as a consequence of Hitchin's Vanishing theorems for positive scalar curvature on $M_1$, as mentioned in the step 5, and $H^1(\mathcal{O}_Q(L^*)) = H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = \mathbb{C}$, as computed in the step 3. This shows that if there is a rational curve of $Q$ on which the Serre class is non-zero, then this class is non-zero and a generator of $H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*))$. The Serre-Horrocks bundle construction on $Z_2$ shows us that $E|_{C_2} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ where $C_2$ is the twistor line on which the blow up is done. We know that $Q = \mathbb{P}_1 \times \mathbb{P}_1 \approx \mathbb{P}(NC_2)$. So that the exceptional divisor has one set of rational curves which are the fibers, and another set of rational curves, coming from the sections of the projective bundle $\mathbb{P}(NC_2)$. Take the zero section of $\mathbb{P}(NC_2)$, on which $E$ has a splitting type $\mathcal{O}(1) \oplus \mathcal{O}(1)$. So over the zero section in $Q$, $E$ is going to be the same, hence non-trivial splitting type. This shows that over this rational curve on $Q$, the Serre-class is nonzero. Hence by the isomorphism above, the Serre-class is the (up to constant) nontrivial class in $H^1(\widetilde{Z}_1, \mathcal{O}(L^*)) \approx \mathbb{C}$.

Next we have to show that this non-trivial class is non-zero on every real twistor line in $\widetilde{Z}_1 - Q$ or $Z_1 - C_1$[1]. For this purpose consider the Serre-Horrocks vector bundle $E_1$ and its section $\zeta_1$ for the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$, so that $\pi^*\zeta_1$ is a section of $\pi^*E_1$ vanishing exactly along $Q$. Remember the construction of the line bundle associated to the divisor $Q$ in $\widetilde{Z}_1$ [GH]. Consider the local defining functions $s_\alpha \in \mathfrak{M}^*(U_\alpha)$[2] of $Q$ over some open cover $\{U_\alpha\}$ of $\widetilde{Z}_1$. These functions are holomorphic and vanish to first order along $Q$. Then the corresponding line bundle is constructed via the transition functions $g_{\alpha\beta} = s_\alpha / s_\beta$. Since $s_\alpha$'s transform according to the transition functions, they constitute a holomorphic section $s$ of this line bundle $[Q]$, which vanish up to first order along $Q$. Local holomorphic sections of this bundle is denoted by $\mathcal{O}([Q])$ and they are local functions with simple poles along $Q$. If we

[1]Thanks to C.LeBrun for this idea.

[2]Here, $\mathfrak{M}^*$ stands for the multiplicative sheaf of meromorphic functions which are not identically zero, in the convention of [GH]. Actually the local defining functions here are holomorphic because Q is effective.

multiply $\pi^*\zeta_1$ with these functions, we will get a holomorphic section of $\pi^*E_1$ on the corresponding local open set, since $\zeta_1$ has a non-degenerate zero on $Q$, so that it vanishes up to degree 1, there. This guarantees that the map is one to one, and the multiplication embeds $\mathcal{O}([Q])$ into $\pi^*E_1$. The quotient has rank 1, and the transition functions of $\pi^*E_1$ relative to a suitable trivialization will then look like

$$\begin{pmatrix} g_{\alpha\beta} & k_{\alpha\beta} \\ 0 & d_{\alpha\beta}\cdot g_{\alpha\beta}^{-1} \end{pmatrix}$$

where $d_{\alpha\beta}$ stands for the determinant of the transition matrix of the bundle $\pi^*E_1$ in this coordinate chart. Since the bundle $\det \pi^*E_1 \otimes [Q]^{-1}$ has the right transition functions, it is isomorphic to the quotient bundle, hence we have the following exact sequence

$$0 \to [Q] \to \pi^*E_1 \to \pi^*K^{-1/2} \otimes [Q]^{-1} \to 0$$

since $\det E_1 = K_{Z_1}^{-1/2}$ as an essential feature of the Serre-Horrocks construction. This extension of line bundles is classified by an element in

$$\mathrm{Ext}^1_{\widetilde{Z}_1}(\pi^*K^{-1/2} \otimes [Q]^{-1}, [Q]) \approx H^1(\widetilde{Z}_1, \pi^*K^{1/2} \otimes [Q]^2)$$

by [AtGr]. If we restrict our exact sequence to $\widetilde{Z}_1 - Q = Z_1 - C_1$, since the bundle $[Q]$ is trivial on the complement of $Q$, this extension class will be the Serre class of the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$. Finally, since $M_1$ has positive scalar curvature, this class is nonzero on every real twistor line in $Z_1 - C_1$. So that non-triviality of the class forced non-triviality over the real twistor lines. In other words $E$ has a non-trivial splitting type over the real twistor lines of $\widetilde{Z}_1$.

So we showed that the Serre-Horrocks vector bundle $E$ determined by $\lambda(\mathcal{P})$ splits as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on all the $\sigma_0$-invariant rational curves in $Z_0$ which are limits of real twistor lines in $\mathcal{Z}_\mathfrak{t}$ as $\mathfrak{t} \to 0$. It therefore has the same splitting type on all the real twistor lines of $\mathcal{Z}_\mathfrak{t}$ for $\mathfrak{t}$ small. Besides,

$$h^j(Z_t, \mathcal{O}(L^*)) \leq h^j(Z_0, \mathcal{O}(L^*)) = 0 \quad \text{for} \quad j = 1, 2$$

by the semi-continuity principle and the proof of the vanishing theorem (5.3). So that via $L^*|_{Z_\mathfrak{t}} \approx K^{1/2}$,

$$H^1(Z_\mathfrak{t}, \mathcal{O}(K^{1/2})) \approx Ker(\Delta + \frac{s}{6}) = 0.$$

Since the two conditions are satisfied, Cohomological characterization (7.3) guarantees the positivity of the conformal class. $\square$

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