Geometric Invariant Theory and Einstein-Weyl Geometry

Mustafa Kalafat

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Abstract

In this article, we give a survey of Geometric Invariant Theory for Toric Varieties, and present an application to the Einstein-Weyl Geometry. We compute the image of the Minitwistor space of the Honda metrics as a categorical quotient according to the most efficient linearization. The result is the complex weighted projective space $\mathbb{CP}_{1,1,2}$. We also find and classify all possible quotients.

1 Introduction

Let (M,g) be a self-dual Riemannian 4-manifold. This means that the anti-self-dual Weyl tensor W_- vanishes. In this case [AHS] construct a complex 3-manifold Z called the *Twistor Space* of M, and a fibration by holomorphically embedded rational curves.

$$\begin{array}{ccll} \mathbb{CP}_1 & \to & Z & \text{Complex 3-manifold} \\ & & \downarrow & \\ & & M^4 & \text{Riemannian 4-manifold} \end{array}$$

Suppose moreover that M admits a free isometric circle(S^1) action. Then the quotient manifold M/S^1 is naturally equipped with a so-called *Einstein-Weyl Geometry*. That is to say we have a triple $(M/S^1,[h],D)$ where $[h]$ is a conformal class, here for the induced metric of the quotient, and D is a torsion-free affine connection. The condition

$$\mathrm{Ric}_{(ij)} = \lambda h_{ij} \qquad \text{(Einstein-like)}$$

more precisely $\mathrm{Ric}(u,v)+\mathrm{Ric}(v,u) = 2\lambda h(u,v)$ and besides that the following

$$Dh = \alpha \otimes h \qquad \text{(Weyl Connection)}$$

for some 1-form α are to be satisfied. This action can naturally be extended to a holomorphic $\mathbb{C}^*$-action over the twistor space. We call the corresponding quotient $Z/\mathbb{C}^*$ the *Minitwistorspace* of the self-dual manifold. It is a very natural question to ask what is this quotient space. We know that if the twistor space is algebraic or Moishezon the quotient becomes a complex surface with singularities in general.

In the march of 2004, Honda gave an explicit description for the twistor space of certain self-dual metrics on $3\mathbb{CP}_2$ admitting a free isometric circle action, equivalently a nowhere zero Killing Field as follows.

Theorem 1.1 (Nobuhiro Honda,[Ho04]). *Let g be a self-dual metric on $3\mathbb{CP}_2$ which admits a non-trivial Killing Field. Suppose further that it is of positive scalar curvature type, and not conformally equivalent to the hyperbolic ansatz self-dual metrics of LeBrun's [Le91].*

Then the twistor space is a small resolution of the double cover of $\mathbb{CP}_3$ branched along a quartic, equation of which is given in some homogeneous coordinates by

$$(Z_2Z_3 + Q(Z_0, Z_1))^2 - Z_0Z_1(Z_0 + Z_1)(Z_0 - aZ_1) = 0$$

where $Q(Z_0, Z_1)$ is a quadratic form of Z_0 and Z_1 with real coefficients, and $a \in \mathbb{R}^+$. Moreover, the naturally induced real structure on $\mathbb{CP}_3$ is given by

$$\sigma(Z_0 : Z_1 : Z_2 : Z_3) = \left(\bar{Z}_0 : \bar{Z}_1 : \bar{Z}_3 : \bar{Z}_2\right),$$

and the naturally induced $U(1)$-action on $\mathbb{CP}_3$ is given by

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \Big(Z_0 : Z_1 : e^{i\theta}Z_2 : e^{-i\theta}Z_3\Big) \quad for \quad e^{i\theta} \in U(1).$$

To construct the minitwistor space of a Honda metric, we appeal to the Geometric Invariant Theory (GIT) for Toric Varieties. This celebrated theory was developed by D. Mumford around 1970's to understand the quotients of group actions on manifolds. We compute the image under the double branched cover, so that we could be able to recover the original minitwistor space by taking a double cover along the related branch locus. GIT computes the quotients according to some linearizations. It takes out some bad orbits, called the unstable orbits and gives a toric variety as a result. We do computations for each linearization and finally figure the way to minimize the number of unstable orbits. Summarizing our main Theorem 4.1 and efficiency arguments in Section 5 we have obtained

Theorem A. The image of the Minitwistor space of a Honda metric in [Ho04] according to some efficient specific linearization is the complex weighted projective space $\mathbb{CP}_{1,1,2}$.

The idea is to compute the coordinate rings of the variety obtained, and sketching the fan or the polytope of the toric variety to realize an isomorphism with the fan or polytope of the $\mathbb{CP}_{1,1,2}$. Yet, one can show that even this most refined quotient excludes $\mathbb{CP}_1$-many orbits. But the GIT for Toric Varieties does not provide a better solution than Theorem A. We define and discuss the efficiency and classification of quotients arising from all possible linearizations. For our purposes, best linearizations are the "efficient" ones as dicussed in Section 5. We are interested in the geometric(visual) perspective, so that reducing the unstable orbits in terms of dimension, measure or number of connected components is desirable for us. Summarizing the Theorems 5.1, 5.2 and Corollary 5.3,

Theorem B. The only possible categorical quotients of $\mathbb{C}^4$ under the $\mathbb{C}^{*2}$ action described by the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ are the empty set, $\mathbb{C}$, $\mathbb{CP}_1$ and $\mathbb{CP}_{1,1,2}$.

Honda also describes these minitwistor spaces in [Ho05] but in a somewhat ad hoc way. So a GIT construction is desirable. The method we use can be applied to compute different minitwistor spaces and also can be developed to be more effective. The project is to apply the general geometric invariant theory and figuring out the complete information about these quotients. It is a very common problem to figure out the minitwistor spaces from the twistor spaces, and there are a number of self-dual metrics for which the minitwistor space is waiting to be computed. A systematic application of GIT will address and solve many problems in the area. This paper should be considered as a modest start for this program. In §2-§3 we give a review of GIT and Toric Varieties. Our survey owe much to the excellent resources [Do, JPB] and [Mu]. Finally in §4-§5 we present our applications.

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2 Action of a torus on an affine space

In this section we will analyze the actions of the algebraic torus group $T = (\mathbb{C}^*)^r$ on the affine space $\mathbb{C}^n$ and understand the quotients arisen this way.

Recall that a *character* χ of an abelian group, with values in a field is a homomorphism from the group to a multiplicative group, i.e. satisfying $\chi(gh) = \chi(g)\chi(h)$. Moreover $\chi(T)$ stands for the group of characters of T. We have the fact that any character $\chi : T \longrightarrow \mathbb{C}^*$ is given by [Do, Mu]

$$\chi(t) = \chi(t_1 \cdots t_r) = t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r} = \prod_{i=1}^{r} t_i^{a_i}$$

for $t_i \in \mathbb{C}$, $a_i \in \mathbb{Z}$. So we have the isomorphism $\chi(T) \approx \mathbb{Z}^r$ for the space of characters.

Consequently, after diagonalization, a T action on $\mathbb{C}^n$ is written as

$$t \cdot \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \chi_1(t) Z_1 \\ \vdots \\ \chi_n(t) Z_n \end{pmatrix} = \begin{pmatrix} t^{a_1} Z_1 \\ \vdots \\ t^{a_n} Z_n \end{pmatrix} = \begin{pmatrix} t_1^{a_{11}} \cdots t_r^{a_{r1}} Z_1 \\ \vdots \\ t_1^{a_{1n}} \cdots t_r^{a_{rn}} Z_n \end{pmatrix},$$

so the matrix $A = [a_{ij}] \in M_{r \times n}(\mathbb{Z})$ encodes the action.

More generally, let $\sigma : T \times X \to X$ be an action of the group T on the complex manifold X by complex automorphisms. For a holomorphic line bundle $\pi : L \to X$, we define

Definition 2.1. *A* linearization *of the holomorphic line bundle L with respect to the action of T is an action $\overline{\sigma} : T \times L \to L$ so that*
(1) The following diagram commutes

$$\begin{array}{ccc} T \times L & \xrightarrow{\overline{\sigma}} & L \\ {\scriptstyle id \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ T \times X & \xrightarrow{\sigma} & X \end{array}$$

(2) The zero section $X \approx L_0 \subset L$ is T- invariant.

So this is the extension of the action σ to L, preserving the fibers, i.e. points on a fiber map onto the same fiber under the action of an element. It follows from the definition that this action on a fiber $\overline{\sigma}_t : L_p \to L_{tp}$ for any $t \in T$ and any $p \in X$ is a linear isomorphism.

In our case, the action of $\mathbb{C}^{r*}$ on $\mathbb{C}^n$ is given by the matrix $A = (a_1 \cdots a_r) \in M_{n \times r}(\mathbb{Z})$. Consider the trivial line bundle $\underline{\mathbb{C}} \to \mathbb{C}^n$. Fix $\alpha = (\alpha_1 \cdots \alpha_r) \in \mathbb{Z}^r$. Extend the action over to the bundle $\underline{\mathbb{C}}$ as follows

$$t \cdot (Z, W) = (t \cdot Z, t^{\alpha} W) = (t \cdot Z, t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r} W) \quad \text{where} \quad Z \in \mathbb{C}^n, W \in \mathbb{C}.$$

We denote this linearized line bundle by L_α. So any $a \in \mathbb{Z}^r$ gives an extension or a linearization.

Recall that the holomorphic sections of the trivial line bundle are identified with the polynomials $F \in \mathbb{C}[Z_1 \cdots Z_n]$, like the homogenous polynomials for bundles over $\mathbb{P}^n$. A section F is an invariant section of L_α if

$$t \cdot (Z, F(Z)) = (t \cdot Z, t^\alpha \cdot F(Z)) = (t \cdot Z, F(t \cdot Z)),$$

which amounts to

$$t^\alpha \cdot F(Z) = F(t \cdot Z),$$

that is

$$t_1^{\alpha_1} \cdots t_r^{\alpha_r} F(Z_1 \cdots Z_r) = F(t^{a_1} Z_1 \cdots t^{a_r} Z_r).$$

The action of $\overline{\sigma}$ on L induces an action on $L^{\otimes d}$ as for a decomposable $l \in L_p^{\otimes d}$,

$$\overline{\sigma}_t(l) = \overline{\sigma}_t(l_1 \otimes \cdots \otimes l_d) = \overline{\sigma}_t(l_1) \otimes \cdots \otimes \overline{\sigma}_t(l_d) \in L_{tp}^{\otimes d}.$$

Likewise, G is an invariant section of $L_\alpha^{\otimes d}$ if for $G = F_1 \cdots F_d$

$$\begin{aligned} G(t \cdot Z) &= F_1(t \cdot Z) \cdots F_d(t \cdot Z) \\ &= (t^\alpha \cdot F_1) \cdots (t^\alpha \cdot F_d) \\ &= t^{\alpha d} \cdot F_1 \cdots F_d \\ &= t^{\alpha d} \cdot G(Z). \end{aligned}$$

Imposing the above condition one proves that

Proposition 2.2 ([Do]). *$G \in H^0(\mathbb{C}^n, L_\alpha^{\otimes d})^T$ i.e. G is an invariant section of the linearized line bundle $L_\alpha^{\otimes d}$ iff it is a linear combination of monomials $Z^m = Z_1^{m_1} \cdots Z_n^{m_n}$ such that*

$$[A \ , -\alpha \ ] \begin{bmatrix} m \\ d \end{bmatrix} = 0_r \qquad \text{(Monomial Equation)}$$

where $A \in M_{r \times n}(\mathbb{Z})$ is the action matrix, $\alpha \in \mathbb{Z}^r$ is the tuple for the extension.

Proof. Say $G = Z^m$, then

$$\begin{aligned} G(t \cdot Z) &= t^{\alpha d} \cdot G(Z) \\ G(t^{a_1} Z_1 \cdots t^{a_n} Z_n) &= (t_1^{\alpha_1} \cdots t_n^{\alpha_n})^d Z^m \\ (t^{a_1} Z_1)^{m_1} \cdots (t^{a_n} Z_n)^{m_n} &= \quad '' \\ t^{a_1 m_1} \cdots t^{a_n m_n} Z^m &= \quad '' \\ (t_1^{a_{11}} \cdot \cdot t_r^{a_{r1}})^{m_1} \cdots (t_1^{a_{1n}} \cdot \cdot t_r^{a_{rn}})^{m_n} Z^m &= \quad '' \end{aligned}$$

Comparing the powers of t_i's from both sides we obtain the equality

$$\begin{aligned} a_{i1} m_1 + \cdots + a_{in} m_n &= \alpha_i d, \\ [a_{i1} \cdots a_{in}] \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} &= [\alpha_i] \, d \quad \text{for any} \ \ 1 \leq i \leq r, \\ Am &= \alpha d. \end{aligned}$$

$\square$

Example 2.3. *Consider the following action of $\mathbb{C}^{*2}$ on $\mathbb{C}^4$,*

$$(t_1, t_2) \cdot \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} t_1 X \\ t_1^{-n} t_2 Y \\ t_1 Z \\ t_2 W \end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The action matrix is $A = \begin{bmatrix} 1 & -n & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ *and the monomials for the invariant sections are obtained from the equation* $\left[\begin{array}{cccc|c} 1 & -n & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right] \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ d \end{bmatrix} = 0_2.$

Next we are going to give some definitions in the Geometric Invariant Theory (GIT), which deals with the actions of groups on manifolds, and figuring out their corresponding quotients.

Definition 2.4 (Stability[Do])**.** *Let L be a T-linearized line bundle on the algebraic variety X and let $x \in X$, then*
(i) x is called semi-stable *with respect to L if it belongs to the set $X \backslash \{s = 0\} \subset \mathbb{C}^n$ (affine) for some $m > 0$ and some $s \in H^0(X, L^m)^T$.*
(ii) x is called unstable *with respect to L if it is not semi-stable.*

We respectively denote by $X^{ss}(L)$ and $X^{us}(L)$, the set of semi-stable and unstable points in X.

Definition 2.5 (Categorical Quotient[Do])**.** *A* categorical quotient *of a T-variety X is a T-invariant morphism $p : X \to Y$ such that for any T-invariant morphism $g : X \to Z$, there exist a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. Y is written sometimes as $X /\!\!/ T$ and also called the* categorical quotient*.*

The GIT guarantees a (good) categorical quotient $X^{ss}(L_\alpha)/T$, see [Do] (pp.118), denoted alternatively by $X(L)/\!\!/_\alpha T$. This is the quotient obtained by taking out the unstable orbits. So according to the GIT, semi-stable points has this well behaving quotient described as follows.

Proposition 2.6 ([Do])**.** *If X is projective and L is ample, we can compute the categorical quotient by*

$$X(L)/\!\!/_\alpha T = \operatorname{Proj}\left(\bigoplus_{d \geq 0} H^0(X, L_\alpha^{\otimes d})^T \right).$$

3 Toric Varieties

Let $V \subset \mathbb{C}^n$ be an affine variety. We define its *(affine) Coordinate ring* to be

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]|_V.$$

This is to say the coordinate ring is the ring of *regular functions* according to the terminology of [Sha]. If we look at the restriction map

$$restr : \mathbb{C}[z_1 \cdots z_n] \longrightarrow \mathbb{C}[z_1 \cdots z_n]|_V$$

we see that its kernel is equal to I_V, the vanishing ideal of V. So the coordinate ring becomes

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]/I_V.$$

For any ring R, we define its *maximal spectrum* by

$$\mathrm{Specm}(R) = \{I < R : I \text{ is a maximal ideal }\}.$$

For any affine variety $V \subset \mathbb{C}^n$, defining the Zariski Topology on each side we have the homeomorphism $V \approx \mathrm{Specm}(\mathbb{C}[V])$ between an affine variety and the maximal spectrum of its coordinate ring. As the trivial case, $\mathbb{C}^n \approx \mathrm{Specm}\,\mathbb{C}[z_1 \cdots z_n]$, where a point $a \in \mathbb{C}^n$ corresponds to its vanishing ideal

$$I_{\{a\}} = \mathbb{C}[z](z_1 - a_1) + \cdots + \mathbb{C}[z](z_n - a_n) = \langle z_1 - a_1, \cdots, z_n - a_n \rangle.$$

The maximal ideals of the latter type consumes the maximal ideals of the polynomial ring $\mathbb{C}[z_1 \cdots z_n]$, see [Mu], which is referred as the Weak Nullstellensatz in the literature [JPB]. The full spectrum is the larger space of prime ideals with which we do not deal here.

For any group G, the *group ring* $\mathbb{C}[G]$ is the vector space with basis $\{[g]\}_{g \in G}$ together with the bilinear product based on group multiplication. This amounts to a $\mathbb{C}$-*algebra*. If we relax the inverse condition on a group then we get similar operations and obtain the *monoid algebra*. As an example we can consider $\mathbb{C}[\mathbb{Z}^n]$ which is the same as the algebra of Laurent polynomials $\mathbb{C}[Z_1^{\pm 1} \cdots Z_n^{\pm n}]$ under the correspondence $m \in \mathbb{Z}^n$ to $Z^m = Z_1^{m_1} \cdots Z_n^{m_n}$. Similarly if we have a submonoid of $\mathbb{Z}^n$, its monoid algebra will be the subalgebra of Laurent polynomials generated by the corresponding monomials. Notice that we are using the same notation with the coordinate ring. The reader is expected to interpret the meaning from the context, will depend on what lies in the bracket, a geometric object or an algebraic one.

We first go into the definition of an affine toric variety. For that purpose we take a cone σ in $\mathbb{R}^n$ satisfying the conditions of the following definition for the canonical lattice $N \approx \mathbb{Z}^n \subset \mathbb{R}^n$.

Definition 3.1 (Cone)**.** *Let $A = \{x_1 \cdots x_r\} \subset \mathbb{R}^n$ be a finite set of vectors. Then*

- *The set $\sigma = \{x \in \mathbb{R}^n : x = \lambda_1 x_1 + \cdots + \lambda_r x_r \ , \ \lambda_i \geq 0\}$ is called a* cone.
- *σ is called a* lattice cone *if all the vectors $x_i \in A$ belong to N.*
- *σ is called* strongly convex *if it does not contain any straight line going through the origin, i.e. $\sigma \cap -\sigma = \{0\}$.*

In this case we define the *affine toric variety* corresponding to σ as

$$U_\sigma := \mathrm{Specm}\,\mathbb{C}[\check{\sigma} \cap N^*]$$

where the dual is defined to be $\check{\sigma} = \{u \in \mathbb{R}^n : \langle u, \sigma \rangle \geq 0\}$ and $N^* = Hom_{\mathbb{Z}}(N, \mathbb{Z})$. By abuse of notation, one can also write $U_\sigma = \mathrm{Specm}\,\mathbb{C}[\check{\sigma}]$. Similar to the way that the cones correspond to an affine toric variety, some collection of cones called fans correspond to a toric variety. More precisely

Definition 3.2 (Fan,[JPB])**.** *A* fan Δ *is a finite union of cones such that*

- *The cones are lattice and strongly convex.*
- *Every face of a cone of Δ is again a cone of Δ.*
- *$\sigma \cap \sigma'$ is a common face of the cones σ and σ' in Δ.*

Now for a fan Δ in N, we can naturally glue $\{U_\sigma : \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytic space

$$X_\Delta := \bigcup_{\sigma \in \Delta} U_\sigma$$

which is irreducible and normal with dimension equal to rank(N) and called the *Toric Variety* [Oda] associated to the fan (N, Δ). It is topologically endowed with an open cover by the affine toric varieties $U_\sigma = \operatorname{Specm} \mathbb{C}[\check{\sigma}]$.

Summarizing what we did in high brow terms [Do], we constructed the $U_\sigma = \operatorname{Specm} \mathbb{C}[\check{\sigma} \cap N^*]$ as the affine variety with $\mathbb{C}[U_\sigma]$ isomorphic to $\mathbb{C}[\check{\sigma} \cap N^*]$. Since for any $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma'$ is a face in both cones, we obtain that $\mathbb{C}[(\sigma \cap \sigma')\check{} \cap N^*]$ is a localization of each algebra $\mathbb{C}[\check{\sigma} \cap N^*]$ and $\mathbb{C}[\check{\sigma}' \cap N^*]$. This shows that $\operatorname{Specm} \mathbb{C}[(\sigma \cap \sigma')\check{} \cap N^*]$ is isomorphic to an open subset of U_σ and $U_{\sigma'}$. Which allows us to glue together the varieties U_σ's to obtain the toric variety X_Δ.

Returning to our case where we have an action of a torus T on an affine space $\mathbb{C}^n$, α-linearized over to a line bundle L, we will now produce a fan and a toric variety out of this linearization. Notice that we have a natural isomorphism of graded algebras

$$\bigoplus_{d \geq 0} H^0(\mathbb{C}^n, L_\alpha^{\otimes d})^T \approx \mathbb{C}[S] = \bigoplus_{d \geq 0} \mathbb{C}[S_d]$$

where S is the monoid of elements $m \in \mathbb{Z}^n$ solving the Monomial Equation. $\mathbb{C}[S_d]$ is the linear span of S_d which is the set of d-th solutions of the Monomial Equation. It is degree-d homogenous part of the finitely generated $\mathbb{C}[S]$. The ideal

$$\mathbb{C}[S]_{>0} := \bigoplus_{d > 0} \mathbb{C}[S_d] = \langle Z^{m_1}, \cdots, Z^{m_s} \rangle$$

is finitely generated by a minimal set of monomial generators where $m_j = (m_{1j} \cdots m_{nj})$. For $I_j := \{i \mid m_{ij} \neq 0\}$ and $Z_I := \Pi_{i \in I} Z_i$ where $I \subset \{1 \cdots n\}$ we have the equality

$$D(Z^{m_j}) := \mathbb{C}^n - \{Z^{m_j} = 0\} = \mathbb{C}^n - \{Z_{I_j} = 0\} =: D(Z_{I_j}).$$

By its definition the semi-stable locus becomes

$$(\mathbb{C}^n)^{ss}(L_\alpha) = \bigcup_{i=1}^{s} D(Z_{I_j}).$$

Thinking the matrix as a map $A : \mathbb{Z}^n \longrightarrow \mathbb{Z}^r$, let $M = \operatorname{Ker} A \subset \mathbb{Z}^n$ and for $1 \leq j \leq s$ define

$$R_j := \mathbb{C}[D(Z_{I_j})]^T = \left\{ \frac{F(Z)}{Z_{I_j}^p} \ : \ p \geq 0 \text{ and } F(Z) \in Z_{I_j}^p \mathbb{C}[M] \right\}.$$

Next, we will be gluing together some affine varieties coordinate rings of which are R_j's. M is a free abelian group of rank $l = n - \operatorname{rank} A$. Consider the map $(\mathbb{Z}^n)^* \longrightarrow N = M^*$, which is given by restricting the linear functionals. Let $\{e_i\}$ be a basis for $\mathbb{Z}^n$, $\{e_i^*\}$ be the dual basis with respect to the Euclidean metric, and $\{\bar{e}_i^*\}$ be their image in M^*. We define the convex cones σ_{I_j}'s or more concisely σ_j's as the following span :

$$\sigma_j := \langle \, \bar{e}_i^* \mid i \notin I_j \, \rangle \subset N_{\mathbb{R}} := N \otimes \mathbb{R} \approx \mathbb{R}^l.$$

One can show that $R_j \approx \mathbb{C}[\check{\sigma}_j \cap M]$. σ_j's form a fan Δ, and this fan gives the toric variety we are seeking as the quotient, see [Do] for details. Consequently we have the folowing,

Theorem 3.3 ([Do]). *Let $(\mathbb{Z}^n)^* \longrightarrow M^*$ be the transpose of the inclusion $M \hookrightarrow \mathbb{Z}^n$ and N be its image. Let Δ be the N-fan formed by the cones $\sigma_j, j = 1 \cdots s$ defined as above. Then*

$$\mathbb{C}^n(L)/\!\!/_{\alpha} T = (\mathbb{C}^n)^{ss}(L_\alpha)/T \approx X_\Delta.$$

Example 3.4. *The weighted projective space $\mathbb{CP}_{1,1,2}$ is by definition the quotient of $\mathbb{C}^3 - 0$ by the $\mathbb{C}^*$-action given by the matrix $A = [1, 1, 2]$.*

If we linearize the trivial bundle over $\mathbb{C}^3$ by $\alpha = 2$, the linear system $Am = \alpha$ is just $a + b + 2c = 2$, and nonnegative solutions for the triple (a, b, c) are generated by

$$(2,0,0) \quad (1,1,0) \quad (0,2,0) \quad (0,0,1)$$

so that the coordinate rings are obtained as

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(2)] = \mathbb{C}[X^2, XY, Y^2, Z]$$

$$\mathbb{C}[U_1/\mathbb{C}^*] = \mathbb{C}[1, \tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{Z}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Z}{X^2}] = \mathbb{C}[a, b]$$

$$\mathbb{C}[U_2/\mathbb{C}^*] = \mathbb{C}[\tfrac{X}{Y}, 1, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[a^{-1}, a, a^{-1}b]$$

$$\mathbb{C}[U_3/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, 1, \tfrac{Z}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Z}{Y^2}] = \mathbb{C}[a^{-1}, ba^{-2}]$$

$$\mathbb{C}[U_4/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}, 1] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}] = \mathbb{C}[b^{-1}, ab^{-1}, a^2b^{-1}]$$

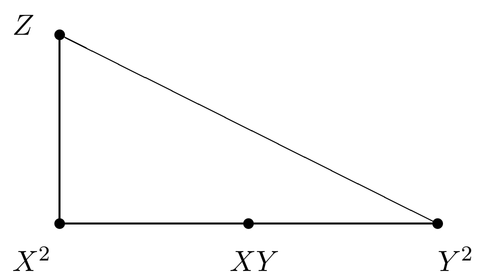
if we assign $a = \frac{Y}{X}$ and $b = \frac{Z}{X^2}$.

Then since

$$\begin{aligned} \textstyle\bigcup_{i=1}^4 U_i &= \mathbb{C}^3 - \{\{X^2 = 0\} \cap \{XY = 0\} \cap \{Y^2 = 0\} \cap \{Z = 0\}\} \\ &= \mathbb{C}^3 - \{\{X = 0\} \cap \{XY = 0\} \cap \{Y = 0\} \cap \{Z = 0\}\} \\ &= \mathbb{C}^3 - \{X = Y = Z = 0\} \end{aligned}$$

these are the coordinate rings of the stated weighted projective space.

The moment polytope looks like :



4 Minitwistor Space

The image of the Honda Minitwistor space (1.1) is the quotient of $\mathbb{CP}_3$ by the $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \big(Z_0 : Z_1 : \lambda Z_2 : \lambda^{-1} Z_3\big) \quad \text{for} \quad \lambda \in \mathbb{C}^*.$$

On the other hand, to obtain $\mathbb{CP}_3$, we already have the classical $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto (\lambda Z_0 : \lambda Z_1 : \lambda Z_2 : \lambda Z_3) \quad \text{for} \quad \lambda \in \mathbb{C}^*.$$

Combining the two, the image equals to the quotient of the $\mathbb{C}^{*2}$ action by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

on $\mathbb{C}^4$. Now, extend this action to the trivial line bundle over $\mathbb{C}^4$. Choices are the linearizations. Among all of them, one of has the minimal number of unstable orbits.

Theorem 4.1. *The categorical quotient of $\mathbb{C}^4$ under the $\mathbb{C}^{*2}$ action described by the matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

linearized by $\alpha = (2, 0)$ is the weighted projective space $\mathbb{CP}_{1,1,2}$.

Proof. The linear system $Am = \alpha$ is

$$\left.\begin{array}{rcl} a+b+c+d & = & 2 \\ c-d & = & 0 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{rcl} a+b+2d & = & 2 \\ c & = & d \end{array}\right.$$

looking for nonnegative solutions, $1, 0$ are the only possibilities for d since from the first equation $2d \leq 2$. So

- $d = 0 : a + b = 2,\ c = 0$ yields the solutions $(2\ \ 0\ \ 0\ \ 0), (1\ \ 1\ \ 0\ \ 0), (0\ \ 2\ \ 0\ \ 0)$.
- $d = 1 : a + b = 0,\ c = 1$ yields the solution $(0\ \ 0\ \ 1\ \ 1)$.

So the coordinate rings are

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(2, 0)] = \mathbb{C}[X^2, XY, Y^2, ZW]$$

$$\mathbb{C}[U_1/\mathbb{C}^{*2}] = \mathbb{C}[1, \tfrac{XY}{X^2}, \tfrac{Y^2}{X^2}, \tfrac{ZW}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{ZW}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{ZW}{X^2}]$$

$$\mathbb{C}[U_2/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2}{XY}, 1, \tfrac{Y^2}{XY}, \tfrac{ZW}{XY}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{ZW}{XY}]$$

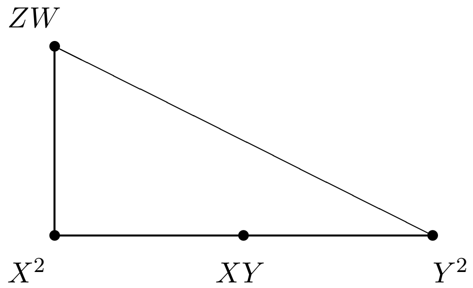
$$\mathbb{C}[U_3/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{XY}{Y^2}, 1, \tfrac{ZW}{Y^2}] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, \tfrac{ZW}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{ZW}{Y^2}]$$

$$\mathbb{C}[U_4/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2}{ZW}, \tfrac{XY}{ZW}, \tfrac{Y^2}{ZW}, 1] = \mathbb{C}[\tfrac{X^2}{ZW}, \tfrac{XY}{ZW}, \tfrac{Y^2}{ZW}]$$

and these coordinate rings are isomorphic to the ones for the $\mathbb{CP}_{1,1,2}$ as in (3.4). Realize the isomorphism by assigning $c = \frac{Y}{X}, d = \frac{ZW}{X^2}$ so that the coordinate rings respectively becomes

$$\mathbb{C}[c, d]\ ,\ \mathbb{C}[c, c^{-1}, c^{-1}d]\ ,\ \mathbb{C}[c^{-1}, c^{-2}d]\ ,\ \mathbb{C}[d^{-1}, cd^{-1}, c^2d^{-1}].$$

Besides, the moment polytope may help to visualize this isomorphism :



□

Realize that the union of U_i's does not cover $\mathbb{C}^4$ since

$$\begin{aligned}\bigcup_{i=1}^4 U_i &= \mathbb{C}^4 - \{\{X^2=0\}\cap\{XY=0\}\cap\{Y^2=0\}\cap\{ZW=0\}\} \\ &= \mathbb{C}^4 - \{\{X=Y=Z=0\}\cup\{X=Y=W=0\}\}.\end{aligned}$$

Consequently, the points $[0:0:0:1],[0:0:1:0]$ in $\mathbb{CP}_3$ are omitted in this quotient.

5 Efficiency and Classification

In this section we analyze the efficiency of the linearization in Theorem 4.1. Our notion of efficiency is based on the maximum dimension of the omitted part under the action which we call the *efficiency dimension.* If this dimension is smaller, we say that the corresponding linearization is more efficient. If two linearizations have the same efficiency dimension, then we consider the measure or number of connected components of the omitted piece to decide which one is more efficient. As an example, the linearization in the theorem has efficiency dimension $\mathrm{Ed}(2,0)=1$.

Theorem 5.1. *Let $\alpha=(x,y)\in\mathbb{N}^2$, i.e. a linearization. Then the following holds.*

- *If $y=0$ then the efficiency dimension $\mathrm{Ed}(0,0)=4$, $\mathrm{Ed}(1,0)=2$, moreover $\mathrm{Ed}(2m,0)=1$ and $\mathrm{Ed}(2m+1,0)=2$ for $m\geq 1$.*
- *If $y\geq 1$ then the efficiency dimension $\mathrm{Ed}(x,y)\geq 3$.*

Proof. Recall that we are considering the following system.

$$\left.\begin{aligned} a+b+c+d &= x \\ c-d &= y\end{aligned}\right\} \text{ or } \left\{\begin{aligned} a+b+2d &= x-y \\ c &= d+y\end{aligned}\right.$$

Since we are concerned with nonnegative solutions, we need to have $x\geq y$ for that purpose. Suppose first that $y=0$.

- $x=0:$ Solution space $SS=\{(0\ 0\ 0\ 0)\}$ is trivial. Charts are empty and $\mathrm{Ed}=4$.
- $x=1:$ $d=0$, $a+b=1$, $c=0$. $SS=\langle(1\ 0\ 0\ 0),(0\ 1\ 0\ 0)\rangle_+$. The quotient turns out to be a $\mathbb{CP}_1$ for the coordinate rings are

  $\mathbb{C}[\mathbb{N}^4\cap\pi^{-1}(1,0)]=\mathbb{C}[X,Y]$

  $\mathbb{C}[U_1/\mathbb{C}^{*2}]=\mathbb{C}[\frac{X}{Y}]=\mathbb{C}[\beta]$

  $\mathbb{C}[U_2/\mathbb{C}^{*2}]=\mathbb{C}[\frac{Y}{X}]=\mathbb{C}[\beta^{-1}]$.

  The omitted locus $\{X=Y=0\}\subset\mathbb{C}^4$ has dimension $\mathrm{Ed}=2$.

- $x = 2m,\ m \geq 1:$ We have $a+b = 2(m-d)$ and $c = d$ in this case. $a+b$ decreases evenly as d increases. So the coordinate ring is as follows.

  $$\mathbb{C}[X^{2m}, X^{2m-1}Y \cdots XY^{2m-1}, Y^{2m}, \{X^{2(m-1)}, X^{2(m-1)-1}Y \cdots Y^{2(m-1)}\}ZW \cdots ZW^m].$$

  This suggests that the ommited locus $X = Y = Z = 0$ or $X = Y = W = 0$, which implies that $\mathrm{Ed} = 1$ for this case.

- $x = 2m+1,\ m \geq 1:$ We have $a+b = 2(m-d)+1$ and $c = d$ in this case. $d \leq m$ for a positive solution to exist.

  The coordinate ring

  $$\mathbb{C}[X^{2m+1}, X^{2m}Y \cdots XY^{2m}, Y^{2m+1}, \{X^{2m-1}, X^{2m-2}Y \cdots Y^{2m-1}\}ZW, \\ \{X^{2m-3} \cdots Y^{2m-3}\}ZW^2 \cdots \{X, Y\}ZW^m]$$

  yields the omitted locus $X = Y = 0$ hence the $\mathrm{Ed} = 2$ in this case.

Now suppose $y \geq 1$. Then from the second equation we have $c = d + y \geq 1$. This tells us that c is nonzero, consequently the hyperplane $Z = 0$ always lies in the ommited locus. □

Theorem 5.2. *Let $\alpha = (x, y) \in \mathbb{N}^2$, i.e. a linearization. Let $y \geq 1$. Then we have the following dimensions and quotients.*

- *If $x - y < 0$ then $\mathrm{Ed} = 4$, and the quotient is empty.*
- *If $x - y = 0$ then $\mathrm{Ed} = 3$, and the quotient is $\mathbb{C}$.*
- *If $x - y = 1$ then $\mathrm{Ed} = 3$, and the quotient is a complex projective line $\mathbb{CP}_1$.*
- *If $x - y \geq 2$ then $\mathrm{Ed} = 3$, and the quotient is the weighted projective space $\mathbb{CP}_{1,1,2}$.*

Proof. The first three cases are similar to that of in the proof of the Theorem 5.1. The rest can be analyzed via splitting into the even and odd cases as

$$x - y = 2m, 2m+1 \ \text{ for } \ m \geq 1.$$

We go over two cases for illustrative purposed, their general cases has the same attributes. If $x - y = 4$ then the coordinate ring can be computed as

$$\mathbb{C}[\{X^4, X^3Y \cdots Y^4\}Z^y, \{X^2, XY, Y^2\}Z^{y+1}W, Z^{y+2}W^2].$$

If $x - y = 3$ then the coordinate ring is

$$\mathbb{C}[\{X^3 \cdots Y^3\}Z^y, \{X, Y\}Z^{y+1}W].$$

We detect the toric variety from the polytopes of these rings as in the Figure 1. The general cases are obtained by extending these polytopes accordingly, which clearly does not change the lattice. A straightforward generalization.

□

Corollary 5.3. *The quotient is the weighted projective space $\mathbb{CP}_{1,1,2}$ for the linearizations in the cases $(2m, 0), (2m+1, 0)$ for $m \geq 1$ of the Theorem 5.1.*

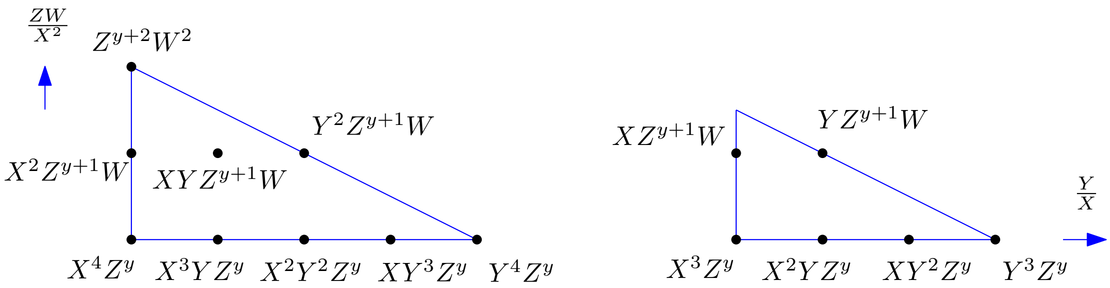


Figure 1: *Typical even*$(x-y=4)$ *and odd*$(x-y=3)$ *cases.*

Proof. The argument of the Theorem 5.2 is still valid in the case of $y = 0$. $\square$

In summary, we compute the minimal efficiency dimension to be equal to 1, and this is achieved by the cases $\alpha = (2m, 0)$ for $m \geq 1$. In all of these minimal cases we have proved that the quotient is the weighted projective space $\mathbb{CP}_{1,1,2}$. We also computed the efficiency and quotients for all the remaining cases.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN AT MADISON
E-mail address : `kalafat@math.wisc.edu`

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