Hyperkähler manifolds with circle actions and the Gibbons-Hawking Ansatz

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Abstract

We show that a complete simply-connected hyperkähler 4-manifold with an isometric triholomorphic circle action is obtained from the Gibbons-Hawking ansatz with some suitable harmonic function.

1 Introduction

A Riemannian Manifold (M, g) is called *Hyperkähler(ian)* if it is Kähler with respect to 3 different complex structures, i.e. three integrable covariantly constant orthogonal complex structures J_1, J_2, J_3 satisfying the quaternionic identity $J_1J_2 = J_3$. We define a *Gravitational Instanton* as a complete Hyperkähler 4-Manifold. By holonomy and dimension reasons a Hyperkähler 4-manifold happens to be Ricci-flat. A motivation to study these type of metrics is that they are exact solutions to the *Einstein field equation* in general relativity

$$\mathrm{Ric} - \tfrac{1}{2}Rg = T. \tag{EFE}$$

The unknowns in the equation are g and T which is the stress-energy(-momentum) tensor, describes the gravitation of the related matter, hence the name. So that a solution consists of a tuple (g, T). If you consider the equation in the vacuum, then T is identically zero because of no gravitation, and taking the trace of both sides yields the extra information $-R = \mathrm{tr}\ T^\sharp$, reducing the (EFE) to $Ric \equiv 0$. That is to say that the Ricci-flat spacetimes describe vacuum solutions of the Einstein's equation. The more general term *Einstein metric* used by mathematicians is again a vacuum solution of the slightly different *Einstein's equation with cosmological constant*.

Next comes the classification problem of the gravitational instantons. In the compact case we have the compact complex surfaces with $c_1 = 0$. Then K3 surfaces and tori T^4 are the only possibilities by the Kodaira classification of compact complex surfaces. In the non-compact case we replace compactness condition with completeness and a type of decay to the flat metric at infinity. There is a classification scheme suggested by Cherkis and Kapustin [Ch] after the work of Gibbons, Hawking, Hitchin, Eguchi, Hanson, Kronheimer, LeBrun, Anderson and others, see [E]. Decompose the 4-manifold with boundary as $M = C \cup N$ such that C is compact and $N = \mathbb{R}^+ \times \partial\overline{M}$. Then suppose that we have a fibration,

$$F \to \partial\overline{M} \to B.$$

Moreover, suppose that we can write the metric asymptotically and locally as $r \in \mathbb{R}^+$

$$g = dr^2 + r^2 g_B + g_F.$$

As $r \to \infty$, base B of the fibration blows up locally in a Euclidean way, volume of the fiber F stays finite and curvature decays to zero which implies that g_F must be flat, F is connected, compact, orientable flat manifold. We have the following cases depending on the dimension of the fiber.

1. $\dim F = 0$: (M, g) *is ALE = Asymptotically Locally Euclidean.* F is a discrete set of points. Complete construction and classification for this case made by P. Kronheimer in 1989 confirming the conjecture of N. Hitchin. Metrics correspond to the finite subgroups $\Gamma < SU(2)$, which are double covers of the finite subroups obtained from $SU(2) \to SO(3)$ except for the odd ordered groups $\mathbb{Z}_{2m+1}$. The constructions use the *Hyperkähler Quotient construction* of Hitchin, Karlhede, Lindström, Roček '87. This is a similar operation to symplectic quotient construction, using three moment maps together. The classification is done using the twistor spaces of the metrics.

   - A_k : *Gibbons-Hawking Multi-Instanton Metrics.* These metrics live on the variety $\mathrm{Res}(\mathbb{C}^\mathbf{2}/\mathbb{Z}_{\mathbf{k+1}})$, which is obtained by resolving $xy - z^{k+1} = 0$. Studied extensively in [GH] and [PG]. The underlying smooth manifolds are obtained by performing plumbing on the matrix,

$$A_k = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}_{k\times k} \quad , \quad \partial M = L(k+1, 1).$$

   Note that $k = 1$ case is the *Eguchi-Hanson(Calabi) Metric* on $\mathcal{O}_{\mathbb{P}_1}(-2) \approx T^*S^2$.

   - $D_k : \Gamma = \mathbb{BD}_{4k}$, Resolve $x^2 + y^2 z + z^{k+1} = 0$.
   - $E_6 : \Gamma = \mathbb{BT}_{24}$, Resolve $x^2 + y^3 + z^4 = 0$.
   - $E_7 : \Gamma = \mathbb{BO}_{48}$, Resolve $x^2 + y^3 + yz^3 = 0$.
   - $E_8 : \Gamma = \mathbb{BI}_{120}$, Resolve $x^2 + y^3 + z^5 = 0$.

2. $\dim F = 1$: (M, g) *is ALF = Asymptotically Locally Flat* and $F \approx S^1$. Conjecturally

   - A_k : *Multi Taub-NUT metrics.* These metrics are studied by Taub'51, Newman-Unti-Tamburino'63, Hawking'77 and LeBrun'91.
   - D_k : Studied by Hitchin'79-Cherkis'08.

3. $\dim F = 2$: (M, g) *is ALG* and $F \approx T^2$. Conjecturally there are two types.

   - $D_k : 0 \leq k \leq 5$.
   - $E_l : 6 \leq l \leq 8$.

4. dim$F = 3$: (M, g) *is ALH* and F is diffeomorphic to one of the six flat orientable closed 3-manifolds. Conjecturally there is only one non-trivial example.

There are also other metrics which are considered or called as Gravitational Instantons by physicist in early 80's. Consult [EGH] for more details. The A_k type ALE and ALF metrics can be explicitly constructed by the *Gibbons-Hawking Ansatz* as follows.

Theorem 1.1 (Gibbons-Hawking[GH]). *Let* $M_0 \longrightarrow \mathbb{R}^3 - \{x_1..x_k\}$ *be the* S^1*-bundle of Chern class* -1 *about each point. Let* V *be the function*

$$V = \epsilon + \frac{1}{2}\sum_{i=1}^{k} \frac{1}{|x - x_i|} \quad , \quad x_i \in \mathbb{R}^3 \ , \ \epsilon \in \mathbb{R}^+.$$

Equip the bundle with the connection one form satisfying $d\omega = *dV$*, a condition which determines the form* ω *uniquely upto gauge. Then the following metric of* M_0

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\omega^2$$

has a smooth completion which is a hyperkähler metric on $M = M_0 \cup \{\tilde{x}_1..\tilde{x}_k\}$.

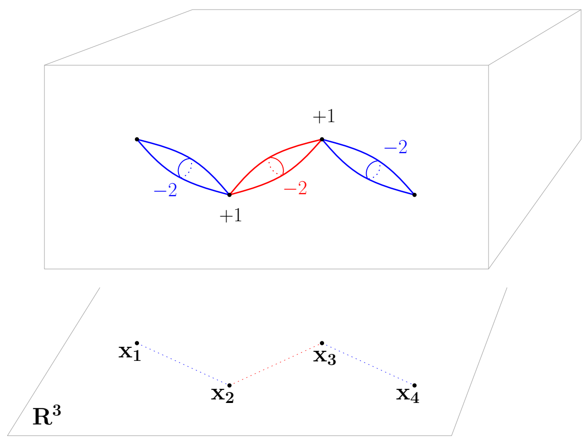

Figure 1: *Inverse image of the line segments between the monopoles are holomorphic curves, generators of the second homology.*

We prove a converse of this in the simply connected case.

Theorem 3.1. *Let* (M, g) *be a complete, simply-connected hyperkähler 4-manifold which admits an isometric, triholomorphic* S^1*-action. Then* g *is obtained from the Gibbons-Hawking ansatz.*

In section §2 we review and study the positive harmonic functions on $\mathbb{R}^n$ with discrete singular points, and in section §3 we prove the main result.

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2 Positive harmonic functions

In this section we classify all positive harmonic functions with isolated singularities on $\mathbb{R}^n$. Our main source is Chapter 3 of [ABR], from which we start by quoting two theorems and a corollary.

Theorem 2.1 (Liouville's Theorem for Positive Harmonic Functions)**.** *A positive harmonic function on $\mathbb{R}^n$ is constant.*

Theorem 2.2 (Bôcher's Theorem)**.** *Suppose u is positive and harmonic on a punctured ball $B\backslash\{0\}$. Then there exists a function v harmonic on B and a constant $b \geq 0$ such that on $B\backslash\{0\}$,*

1. $u(x) = b\log(1/|x|) + v(x)$ *(if $n = 2$),*
2. $u(x) = b|x|^{2-n} + v(x)$ *(if $n > 2$).*

Corollary 2.3. *Suppose $n > 2$. If u is positive and harmonic on $\mathbb{R}^n\backslash\{0\}$, then there are nonnegative constants b and c such that $u(x) = b|x|^{2-n} + c$ for all $x \in \mathbb{R}^n\backslash\{0\}$.*

Next we consider positive harmonic functions with multiple singularities; the characterization of such functions is posed as Exercise 17 in Chapter 3 of [ABR].

Proposition 2.4. *Suppose $n > 2$. Let $A = \{a_1, a_2, \ldots\}$ denote a discrete subset of $\mathbb{R}^n$. If u is positive and harmonic on $\mathbb{R}^n\backslash A$, then there are nonnegative constants b_i and c such that*

$$u(x) = \sum_{i=1}^{\infty} b_i|x - a_i|^{2-n} + c$$

for all $x \in \mathbb{R}^n\backslash A$.

Remark 2.5. *The convergence of the above sum implies, a posteriori, that there must be adequate spacing between the points a_i.*

Proof. Let B_i be a ball centred at a_i which is sufficiently small that it does not contain any other a_j. By Bôcher's theorem there is a harmonic function v_i on B_i and a constant $b_i \geq 0$ such that

$$u(x) = b_i|x - a_i|^{2-n} + v_i(x)$$

on B_i. First we claim that

Lemma 2.6.

$$w(x) = \sum_{i=1}^{\infty} b_i|x - a_i|^{2-n} \qquad \text{converges.}$$

Proof. This follows from Harnack's principle [ABR]: a pointwise increasing sequence of harmonic functions on a connected region Ω must either diverge to infinity at every point or converge uniformly on compact subsets to a harmonic function on Ω. We apply this to the sequence

$$u_m(x) = \sum_{i=1}^{m} b_i|x - a_i|^{2-n}$$

with $\Omega = \mathbb{R}^n \backslash A$. The singularities $\{a_1, \dots, a_m\}$ of $u(x) - u_m(x)$ are removable, and hence $u(x) - u_m(x)$ extends to a harmonic function on $\mathbb{R}^n \backslash \{a_{m+1}, a_{m+2}, \dots\}$. Since $u(x)$ is positive and $u_m(x) \to 0$ at infinity, we have

$$\liminf_{x \to \infty} (u(x) - u_m(x)) \geq 0.$$

For $i \geq m+1$ either b_i is zero or strictly positive. We can effectively ignore the former case: if $b_i = 0$ then $u(x)$ has no singularity at a_i and we could have removed a_i from A at the outset. If $b_i > 0$ then $u(x) \to \infty$ at a_i and so

$$u(x) - u_m(x) \gg 0$$

on $B_i' \backslash \{0\}$ for some sufficiently small ball $B_i' \subset B_i$. Applying the minimum principle to $u(x) - u_m(x)$ on $\mathbb{R}^n \backslash \bigcup_{i=m+1}^{\infty} B_i'$ we conclude that $u(x) - u_m(x)$ is nonnegative. Therefore $u_m(x)$ is bounded above by $u(x)$, and the sequence $u_m(x)$ cannot diverge to infinity. By Harnack's principle $u_m(x)$ converges uniformly on compact subsets to a harmonic function $w(x)$ on $\mathbb{R}^n \backslash A$. □

Next we define $v(x) = u(x) - w(x)$ for $x \in \mathbb{R}^n \backslash A$. The singularities at a_i are removable and $v(x)$ extends to a harmonic function on $\mathbb{R}^n$. We already saw that $u(x) - u_m(x)$ is nonnegative; taking the limit as $m \to \infty$ shows that $v(x) = u(x) - w(x)$ is also nonnegative. By Liouville's theorem $v(x)$ is constant, and this completes the proof. □

3 Gibbons-Hawking Ansatz

Theorem 3.1. *Let (M, g, J_1, J_2, J_3) be a complete, simply-connected hyperkähler 4-manifold. Let M admits an isometric, triholomorphic S^1 action. Then g is obtained from the Gibbons-Hawking ansatz.*

Proof. Let $\Omega_i = g(J_i., .)$ be the corresponding Kähler forms for $i = 1, 2, 3$. Since M is simply-connected $H_1(M, \mathbb{R})$ is trivial, and by the universal coefficients theorem this implies that $H^1_{dR}(M)$ is also trivial. So the closed one forms are exact on M. Let X be the Killing field generated by the action of S^1, i.e.

$$X_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot \exp(t1) \quad \text{where} \quad p \in M, \ 1 \in \mathfrak{u}(1) = \mathbb{R}.$$

Since the flow of X preserves g and J_i it also preserves Ω_i. Moreover Ω_i is closed so we can use the Cartan formula for the Lie derivative of the form by the vector to show that $i_X \Omega_i$ is a closed one form. By above it is exact so that we have functions $\mu_i : M \to \mathbb{R}$ such that

$$i_X \Omega_i = -d\mu_i \quad \text{for} \quad i = 1, 2, 3.$$

Lemma 3.2. *The orbits of the S^1 action are circles and isolated points.*

Proof. Since X is holomorphic with respect to three complex structures, its vanishing locus should be holomorphic with respect to these complex structures. This locus cannot be 2 real dimensional since a surface cannot be holomorphic according to all three complex structures, different complex structures will send a vector in the tangent plane of a surface to different vectors some of which leaves the tangent plane. Therefore the vanishing locus is either 0 or 4 dimensional. Because of the analytic continuation we eliminate the 4 dimensional case so the vanishing locus is 0 dimensional.

Suppose that there is an accumulation point for the zeros of X, i.e. singular points of the action. Take a neighborhood of the accumulation point which has a holomorphic basis v_1, v_2 with respect to the first complex structure. Since X is Killing and holomorphic it is the real part of the holomorphic vector field $X_{\mathbb{C}} = X - iJ_1X$. Then $X_{\mathbb{C}} = fv_1 + gv_2$ for some holomorphic functions f, g. Zero locus of a generic holomorphic function is codimension one, so at least one of these two functions say f vanishes along a complex curve C_f. Now consider the restriction $g : C_f \to \mathbb{C}$. It vanishes on a set with an accumulation point. Therefore one dimensional holomorphic function theory [C] implies that g is identically zero on the curve C_f. X has to vanish there as well which is a contradiction. ☐

Lemma 3.3. *$d\mu_1, d\mu_2, d\mu_3, X^\flat$ are linearly independent whenever $X \neq 0$.*

Proof. Pick a point $p \in M$ where $X_p \neq 0$. Let

$$c_1 d\mu_1 + c_2 d\mu_2 + c_3 d\mu_3 + c_4 X^\flat = 0.$$

Feeding X to both sides yields $c_4|X|^2 = 0$, hence $c_4 = 0$.
Feeding any vector field Y to the rest we have

$$\begin{aligned} 0 &= \sum_{i=1}^{3} c_i d\mu_i(Y) \\ &= \sum_{i=1}^{3} c_i \Omega_i(X, Y) \\ &= g\left(\sum_{i=1}^{3} c_i J_i X, Y\right). \end{aligned}$$

Since this holds for arbitrary Y by non-degeneracy we have

$$\sum_{i=1}^{3} c_i J_i X = 0.$$

Applying the operator $\Sigma_{i=1}^3 c_i J_i$ both sides gives $-\Sigma_{i=1}^3 c_i^2 = 0$ since the mixed terms cancel, consequently $c_i = 0$ for $i = 1, 2, 3$. ☐

This in particular implies that the map $\mu : M \to \mathbb{R}^3$, $\mu = (\mu_1, \mu_2, \mu_3)$ obtained by combining the three maps has rank 3, hence a submersion away from the singular points. Thus μ is an open map here.

Next we would like to show that the map μ is surjective. For this purpose we first want to make it into a Riemannian submersion out of the singular fibers by furnishing $\mathbb{R}^3$ with the right metric h. Define the scalar function $V := 1/g(X, X)$ on $M - \{sing\}$. This descends to a function V on $\mathbb{R}^3 - \{sing\}$ since the flow of the Killing field keeps the norm invariant.

We have $g(J_iX, J_jX) = \delta_{ij}V^{-1}$ taking $J_0 = Id$, so $\{X, J_1X, J_2X, J_3X\}$ is an orhogonal basis. Since the common trivial locus of $d\mu_i$ is $\langle X\rangle$, the orhogonal complement becomes $Ker d\mu^{\perp} = \langle J_1X, J_2X, J_3X\rangle$. We will denote this distribution by $\mathcal{H} = \cup_{p\in M-\{sing\}}\mathcal{H}_p$. Since

$$d\mu\left(\sum_{i=1}^{3} a^i J_i X\right) = -V^{-1}\sum_{i=1}^{3} a^i \frac{\partial}{\partial \mu^i}$$

we need to have

$$h_{ij} = h\left(\frac{\partial}{\partial\mu^i}, \frac{\partial}{\partial\mu^j}\right) = g(VJ_iX, VJ_jX) = V\delta_{ij}.$$

This ensures the following

Proposition 3.4. *The map* $\mu : (M, g) \to (\mathbb{R}^3, h)$ *where*

$$h = V((d\mu^1)^2 + (d\mu^2)^2 + (d\mu^3)^2)$$

is a Riemannian submersion away from the fixed points.

Proposition 3.5. *The map* $\mu : M \to \mathbb{R}^3$ *is surjective.*

Proof. The completeness of M will be the main ingredient of the proof. Pick a point $b \in \mathbb{R}^3$ which is out of the range of μ. Join this to a point a in the image with a line segment $\overline{ab}$ which does not pass through any image of the singular points. Let c be the point on $\overline{ab}$ which achieves the supremum of $\{|ae| : \overline{ae} \subset \overline{ab}\cap\mu(M)\}$. $c \notin \mu(M)$ because μ being a submersion would imply that c is not the infimum. Since $\overline{ab}$ does not pass through any singular points, we can lift the segment $\overline{ac}[$ using an ODE system as a horizontal curve. This horizontal curve has the same finite length as of $\overline{ac}[$. By completeness the Cauchy sequences heading above c should converge and forces $c \in \mu(M)$ a contradiction. □

Next we will write down the metric on M explicitly in local coordinates. We already have an orhogonal basis away from the singularities so that we can write

$$g = V^{-1}((X^*)^2 + (J_1X^*)^2 + (J_2X^*)^2 + (J_3X^*)^2).$$

Locally around a nonsingular point we can write $X^\flat = d\theta$ for some real valued function θ. Because of the linear independence (3.3), $(\mu_1, \mu_2, \mu_3, \theta)$ defines a local coordinate system.

Let us write our orhogonal basis in terms of these local coordinates. For this purpose we shall first need a one form. Let $\omega \in \Omega^1(M - \{sing\})$ be the one-form which has $Ker(\omega) = \mathcal{H}$ and $\omega(\partial/\partial\theta) = 1$ so that it is uniquely defined. This is actually the connection one form of the 3-dimensional distribution $\mathcal{H}$. Let $\alpha = \omega - d\theta$ and $\alpha_i = \alpha(\partial/\partial\mu^i)$. Now we can proceed. The equalities

$$\mu_*\left(\frac{\partial}{\partial\mu^i}\right) = -\frac{\partial}{\partial\mu^i} \quad \text{and} \quad \mu_*\left(J_i\frac{\partial}{\partial\theta}\right) = V^{-1}\frac{\partial}{\partial\mu^i}$$

implies that

$$V^{-1}\frac{\partial}{\partial\mu^i} + J_i\frac{\partial}{\partial\theta} = f\frac{\partial}{\partial\theta}$$

for some function f, since the left-hand side does not have any μ component. Applying ω to both sides of this equality yields $V^{-1}\alpha_i = f$ since $J_i\partial/\partial\theta$ is horizontal. So we have

$$J_i\frac{\partial}{\partial\theta} = -V^{-1}\left(\frac{\partial}{\partial\mu^i} - \alpha_i\frac{\partial}{\partial\theta}\right) \quad \text{for} \quad i = 1, 2, 3,$$

where

$$\frac{\hat{\partial}}{\partial\mu^i} = \frac{\partial}{\partial\mu^i} - \alpha_i\frac{\partial}{\partial\theta}$$

is the lift of $\partial/\partial\mu^i$ to $\mathcal{H}$ since it maps to $\partial/\partial\mu^i$ and is perpendicular to $\partial/\partial\theta$. Accordingly in local coordinates we have the orthonormal basis

$$\{V^{-1/2}\frac{\hat{\partial}}{\partial\mu^1}, V^{-1/2}\frac{\hat{\partial}}{\partial\mu^2}, V^{-1/2}\frac{\hat{\partial}}{\partial\mu^3}, V^{1/2}\frac{\partial}{\partial\theta}\}.$$

The components of the metric becomes

$$g = \begin{pmatrix} \frac{V^2+\alpha_1^2}{V} & \frac{\alpha_1\alpha_2}{V} & \frac{\alpha_1\alpha_3}{V} & \frac{\alpha_1}{V} \\ \frac{\alpha_2\alpha_1}{V} & \frac{V^2+\alpha_2^2}{V} & \frac{\alpha_2\alpha_3}{V} & \frac{\alpha_2}{V} \\ \frac{\alpha_3\alpha_1}{V} & \frac{\alpha_3\alpha_2}{V} & \frac{V^2+\alpha_3^2}{V} & \frac{\alpha_3}{V} \\ \frac{\alpha_1}{V} & \frac{\alpha_2}{V} & \frac{\alpha_3}{V} & \frac{1}{V} \end{pmatrix},$$

so that this is exactly the matrix of the following tensor

$$g = V((d\mu^1)^2 + (d\mu^2)^2 + (d\mu^3)^2) + V^{-1}\omega^2$$

where $\omega = d\theta + \alpha$.

Theorem 3.6. *The induced map* $V : \mathbb{R}^3 - \{sing\} \to \mathbb{R}^+$ *is harmonic.*

Proof. Since $\mathcal{L}_X g = 0$, X has constant length along the fibers, hence V is constant along the circle fibers, descents to a function on $\mathbb{R}^3 - \{sing\}$.

The idea is to analyse the Hyperkähler structure. For the first complex structure we have

$$J_1 \ : \ V^{-1/2}\frac{\hat{\partial}}{\partial\mu^1} \mapsto V^{1/2}\frac{\partial}{\partial\theta} \quad , \quad V^{-1/2}\frac{\hat{\partial}}{\partial\mu^2} \mapsto V^{-1/2}\frac{\hat{\partial}}{\partial\mu^3}.$$

The second map is a consequence of the fact that $\hat{\partial}/\partial\mu^2, \hat{\partial}/\partial\mu^3$ are in the orthogonal complement of the $\hat{\partial}/\partial\mu^1, \partial/\partial\theta$. So that they span an invariant plane. The choice of the orientation determines the action uniquely on the plane. The other two complex structures has similar descriptions, and when we dualize we obtain

$$J_i : Vd\mu^i \mapsto \omega \ , \ d\mu^{i+1} \mapsto d\mu^{i+2} \ \ (mod\ 3) \ \text{ for } \ i = 1, 2, 3.$$

Kähler form of the first complex structure is computed as

$$\Omega_1 = d\mu^1 \wedge \omega + Vd\mu^2 \wedge d\mu^3.$$

Since the metric is Kähler with respect to this complex structure it is closed,

$$\begin{aligned}0 &= d\Omega_1\\ &= d\mu^1 \wedge (-d\alpha + V_1\, d\mu^2 \wedge d\mu^3)\\ &= (\alpha_{2,3} - \alpha_{3,2} + V_1)\, d\mu^1 \wedge d\mu^2 \wedge d\mu^3,\end{aligned}$$

where $\alpha_{i,j} = \partial\alpha_i/\partial\mu^j$ and $V_i = \partial V/\partial\mu^i$. Combining the other two conditions

$$\alpha_{3,2} - \alpha_{2,3} = V_1$$

$$\alpha_{1,3} - \alpha_{3,1} = V_2$$

$$\alpha_{2,1} - \alpha_{1,2} = V_3.$$

Hence

$$\mathrm{Curl}\,\alpha = \mathrm{Grad}\,V.$$

In terms of the Hodge star operator this is *the Bogomolny equation*

$$d\alpha = *\,dV.$$

□

From the analysis in section §2, by the Proposition 2.4 we conclude that for nonnegative constants b_i, c and for $x \in \mathbb{R}^3 - \{sing\}$, we have

$$V(x) = c + \sum_{i=1}^{\infty} \frac{b_i}{|x - a_i|}$$

where the singular locus $\{sing\} = \{a_1, a_2, \ldots\}$.

Next we will look for the resctrictions on the coefficients. The connection one form α and the function V are related to each other by the Bogomolny equation. The curvature of the circle bundle is given by $d\alpha$, so by Chern-Weil theory the first Chern class of the circle bundle is computed as

$$c_1(M) = [\frac{1}{2\pi} d\alpha] \in H^2(\mathbb{R}^3 - \{sing\}, \mathbb{Z}).$$

Here what we mean by the integral cohomology is the cohomology classes which give integer values on the generators of the second integral homology. Since $\mathbb{R}^3 - \{sing\}$ deformation retracts on a wedge of spheres, generators of the integral homology $H_2(\mathbb{R}^3 - \{sing\}, \mathbb{Z})$ are small spheres centered at the singular points, and we expect the integral of the first Chern class on these spheres to be integer. Furthermore, the following fact puts restriction on the coefficients.

Fact 3.7. *In order for the compactified 4-manifold to be smooth, first Chern class of the circle bundle must be -1 around each missing point.*

This is a topological reason. The boundary of a small ball neighborhood of a point in a 4-manifold is diffeomorphic to an S^3. So, whenever completing an incomplete 4-manifold by gluing in a point, the gluing boundary should be diffeomorphic to S^3. Otherwise the glued in point will cause a singularity. In our situation the boundary of a small ball around the missing point is diffeomorphic to a circle bundle over S^2. These bundles are classified by the integers: the bundle is trivial over the north and south hemispheres of S^2, so it is determined by the homotopy type of the gluing map, which is a map from the equator S^1 to the fibre S^1. Up to homotopy, maps from S^1 to S^1 are determined by their degree in $\mathbb{Z}$. Degree zero corresponds to the trivial circle bundle $S^1 \times S^2$. Degree -1 corresponds to S^3, the Hopf fibration. Other degrees produce three-manifolds other than S^3. A related fact is that contracting -1-curves yields a smooth variety. Also a Chern class $-k$ bundle gives an orbifold point related to the cyclic group $\mathbb{Z}_k$. See also [Kr].

Let us compute a first Chern class around the i-th singular point, taking the radius ρ of the sphere small enough so that no other singular points are in. Terms of V other than the i-th one will not contribute to the integral because the sphere misses their related points. Taking the i-th point to be the origin for the convenience since the Lebesgue measure is translation invariant and using the spherical coordinates we obtain,

$$\begin{aligned}
-1 &= \langle c_1(M), [S^2_\rho] \rangle \\
&= \int_{S^2_\rho} \frac{1}{2\pi} * dV \\
&= \int_{S^2_\rho} \frac{1}{2\pi} \frac{-b_i}{|(\mu^1, \mu^2, \mu^3)|^3} (\mu^1 d\mu^2 \wedge d\mu^3 + \mu^2 d\mu^3 \wedge d\mu^1 + \mu^3 d\mu^1 \wedge d\mu^2) \\
&= \frac{b_i}{2\pi} \int_{S^2_\rho} -\sin\phi \, d\phi \wedge d\theta \\
&= \frac{b_i}{2\pi}(-4\pi) \\
&= -2b_i.
\end{aligned}$$

So that $b_i = 1/2$ for all $i \in \mathbb{Z}^+$. Consequently we have the following.

Corollary 3.8. *For a constant $c \geq 0$ and for $x \in \mathbb{R}^3 - \{sing\}$, we have*

$$V(x) = c + \frac{1}{2} \sum_{i=1}^{\infty} \frac{1}{|x - a_i|}$$

where the singular locus $\{sing\} = \{a_1, a_2, \ldots\}$.

$\square$

Remark 3.9. *Notice that for $c = 0$ the metrics we have are ALE, and for $c > 0$ the metrics we have are ALF. Moreover there is only one metric corresponding to case $c > 0$ upto homothety as follows.*

$$\begin{aligned}
g_c &= (c + \tfrac{1}{2} \textstyle\sum \frac{1}{|x-a_i|}) dx^2 + (c + \tfrac{1}{2} \sum \frac{1}{|x-a_i|})^{-1} \omega^2 \\
&= c^{-1}((1 + \tfrac{1}{2} \textstyle\sum \frac{1}{|cx-ca_i|}) d(cx)^2 + (c + \tfrac{1}{2} \sum \frac{1}{|cx-ca_i|})^{-1} \omega^2) \\
&= c^{-1}((c + \tfrac{1}{2} \textstyle\sum \frac{1}{|y-a_i|}) dy^2 + (c + \tfrac{1}{2} \sum \frac{1}{|y-a_i|})^{-1} \omega^2) \\
&= c^{-1} g_1.
\end{aligned}$$

Rescaling x and the a_i by c is an isometry, multiplying the metric by a constant is a homothety. Since we did not change the θ coordinate, and ω depends on the distribution and $\partial/\partial\theta$, the ω^2 term is not affected by these operations. This shows that any metric of non-zero type is homothetic to g_1, the metric corresponding to $c = 1$.

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