

Free immersions and panelled web 4-manifolds

Mustafa Kalafat

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Abstract

We show that if a compact, oriented 4-manifold admits a coassociative($*\phi_0$)-free immersion into \mathbb{R}^7 then its Euler characteristic χ_M and signature τ_M vanish. Moreover, in the spin case the Gauss map is contractible, so that the immersed manifold is parallelizable. The proof makes use of homotopy theory in particular obstruction theory. As a further application we prove a non-existence result for some infinite families of 4-manifolds that can not be addressed previously. We give concrete examples of parallelizable 4-manifolds with complicated non-simply-connected topology.

1 Introduction

Let (M, g) be a Riemannian n -manifold. If we take a point p in M , a k -dimensional vector subspace $V < T_p M$ equipped with an orientation is called an *oriented tangent k -plane* of M . In this case the restricted metric $g|_V$ and the orientation gives a natural k -form, the volume form vol_V on V . A k -form φ on M is called a *calibration* if it is closed and $\varphi|_V \leq \text{vol}_V$ for any oriented k -plane V . In general $\varphi|_V = \alpha \cdot \text{vol}_V$ for some $\alpha \in \mathbb{R}$ since they are both top forms on the vector space V . The calibration condition is equivalent to $\alpha \leq 1$. Under these assumptions if N is an oriented k -dimensional submanifold of M , then the tangent spaces of N are automatically oriented tangent k -planes and we say that N is a *calibrated submanifold* or *φ -submanifold* of M if N has maximal tangent spaces i.e. $\varphi|_{T_p N} = \text{vol}_{T_p N}$ for all $p \in N$. The function $\alpha \equiv 1$ constant on N in this case. Calibrated manifolds are introduced in [HL82], for a survey see also [Joy00]. On the Euclidean space \mathbb{R}^7 , consider the *coassociative form* which is the 4-form

$$*\phi_0 = dx^{1234} - dx^{12-34} \wedge dx^{67} - dx^{13-42} \wedge dx^{75} - dx^{14-23} \wedge dx^{56}.$$

This is actually an example of a calibration. The subgroup of $GL(7, \mathbb{R})$ that leaves $*\phi_0$ invariant is the compact 14-dimensional Lie group G_2 . If U is the 4-plane in \mathbb{R}^7 with last three coordinates vanishing, then $*\phi_0|_U = dx^{1234}$ which is equal to vol_U

with suitable orientation on U hence U is a maximal plane called a *coassociative 4-plane*. One can show that [HL82] the subgroup of G_2 preserving U is $SO(4)$, and G_2 acts transitively on (oriented) coassociative 4-planes. So that the set COASS of coassociative planes in \mathbb{R}^7 is isomorphic to $G_2/SO(4)$ and has dimension 8. The Grassmannian $G_4^+\mathbb{R}^7$ of all oriented 4-planes in \mathbb{R}^7 is of dimension 12 so COASS is a codimension 4 subspace of it. We also have that $\phi_0|_V = 0$ for every coassociative 4-plane V since $\phi_0|_U = 0$ by definition, the action of G_2 is transitive on the coassociative 4-planes and ϕ_0 is G_2 invariant. Conversely if $\phi_0|_V = 0$ for a 4-plane then there is a unique orientation which makes V a coassociative 4-plane.

We are actually interested in the following type of submanifolds. Following [HLog] if (M, ϕ) is a calibrated manifold, a submanifold N is called ϕ -free if there are no ϕ -planes tangent to N . These submanifolds have strictly ϕ -convex neighborhoods each of which admits deformation retraction onto N . These are generalizations of totally real submanifolds in Kähler manifolds to calibrated manifolds, and strictly ϕ -convex manifolds are the generalization of Stein manifolds. They have nice topological structures. In order to understand ϕ -free submanifolds, some special information is needed about the related Grassmann manifolds. We will be using the results of [AK16] on the topology of Grassmannians. As an application we will give an answer to coassociative-free embedding problems for some infinite families of 4-manifolds. There is very few results on the coassociative-free embeddings of 4-manifolds. Only notable obstruction is the Euler characteristic of İ. Ünal in [Ü11]. One can for example conclude that coassociative-free embeddings of the 4-manifolds $S^4, \Sigma_g \times \Sigma_h$ for genus $g, h \neq 1$ are violated since they have non-zero Euler characteristic. On the other hand, the techniques in the literature can not answer this question for $\chi = 0$ case. Due to our main result in this paper, we now able find better obstructions as follows.

Theorem 2.6.(Vanishing). *If M^4 is closed and $i : M \rightarrow \mathbb{R}^7$ is a coassociative($\ast\phi_0$)-free immersion, then the Euler characteristic χ_M and the signature τ_M vanishes.*

If we combine it with the results in [Ü15] on the converse, we obtain the following.

Corollary 1.1. *A closed 4-manifold M admits a coassociative($\ast\phi_0$)-free immersion or embedding into \mathbb{R}^7 if and only if its Euler characteristic χ_M and the signature τ_M vanishes.*

One can also change the target space from the Euclidean space to any manifold with G_2 structure which is flat in a neighborhood of a point.

In the spin case we are now able to give a better obstruction as follows. We will make use of the fact that in dimension four, parallelizability is characterized through the following complete obstructions.

Theorem 1.2 ([HH58, Mas58]). *A smooth 4-manifold is parallelizable iff $w_{12} = e = p_1 = 0$.*

See also [Tho68]. So this implies that, further in the spin case the 4-manifold has to be parallelizable. We alternatively prove this fact going through the analysis of the Gauss map, and show that it is trivializable as well.

Theorem 4.4. *If M^4 is a compact, oriented, spin manifold and $i : M \rightarrow \mathbb{R}^7$ is a coassociative($\ast\phi_0$)-free immersion, then its image $g(M)$ under the Gauss map $g : M \rightarrow G_4^+\mathbb{R}^7$ is contractible, so that M is parallelizable.*

Our Proposition 2.5 is crucially used in the proof of the main result of the paper [Ü15]. So our paper fills a gap in the literature in this perspective. As another application we illustrate our results through a series of examples in section §5. We do a similar computation from this series in section §3 for the Cayley-free case. In [Ü18] using h-principle techniques, an if and only if theorem is proved for this case as well. In section §3 we prove our main result, in section §4 we prove an injectivity lemma and applications on the Gauss map.

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2 Coassociative-free Immersions

In this section we will present applications to the immersion theory. We will be using results on the topology of the oriented Grassmann space $G_3^+\mathbb{R}^7$. We denote the canonical(tautological) vector bundle and its orthogonal complement by $E = E_3^7$ and $F = F_3^7$ on this space. We will often be using the following result.

Theorem 2.1 ([SZ14]). *We have the following characteristic class relations for the bundles over the Grassmannian $G_3^+\mathbb{R}^7$,*

(a) $p_1E = -p_1F, p_1^2E = e^2F.$

(b) $p_1E[\mathbb{C}P_2] = p_1E[\overline{\mathbb{C}P}_2] = eF[\mathbb{C}P_2] = -eF[\overline{\mathbb{C}P}_2] = 1.$

(c) $\frac{1}{2}(p_1E \pm eF)$ are generators in $H^4(G_3^+\mathbb{R}^7; \mathbb{Z})$. Their Poincaré duals are

$[ASS]$ and $[\widetilde{ASS}]$ respectively.

(d) $\frac{1}{2}(p_1EeF \pm e^2F)$ are generators in $H^8(G_3^+\mathbb{R}^7; \mathbb{Z})$. Their Poincaré duals are

$[\mathbb{C}P]$ and $[\overline{\mathbb{C}P}_2]$ respectively.

This is a combination of the results in Section 7 of the resource. Notice that, to be able to say that the generators stated in part (c) are the sole generators, one needs to know that there is no torsion as explained in [AK16]. Also note that we can realize the embeddings of $\mathbb{C}P_2$ and $\overline{\mathbb{C}P}_2$ through the inclusion of $G_2^+\mathbb{R}^6$ in $G_3^+\mathbb{R}^7$ since the oriented Grassmannian is a double cover of the Grassmannian,

we can include the projective space with both orientations. These two projective spaces are the generators of $H_4(G_3^+\mathbb{R}^7; \mathbb{R})$ as explained in the resource.

Let $i : M \rightarrow \mathbb{R}^7$ be an immersion of a 4-manifold into the Euclidean space ¹. In this case we have the associated Gauss map $g : M \rightarrow G_4^+\mathbb{R}^7$ from the manifold to the oriented Grassmannian. It sends a point to the 4-dimensional subspace of the Euclidean space which is parallel to the tangent space at that point. In general no such map exist due to lack of translation unless e.g. the space is flat and simply connected or alternatively parallelizable. Composing this map with the orthogonal complement map $* : G_4^+\mathbb{R}^7 \rightarrow G_3^+\mathbb{R}^7$ we get the map

$$\tilde{g} : M^4 \longrightarrow G_4^+\mathbb{R}^7 \longrightarrow G_3^+\mathbb{R}^7 \quad \text{where} \quad \tilde{g} = * \circ g$$

which is appropriate setting for us to apply the results above. Keep in mind that the orthogonal complement map the pullbacks the bundles as $*^*E_3^7 = F_4^7$ and $*^*F_3^7 = E_4^7$. We start with our first lemma.

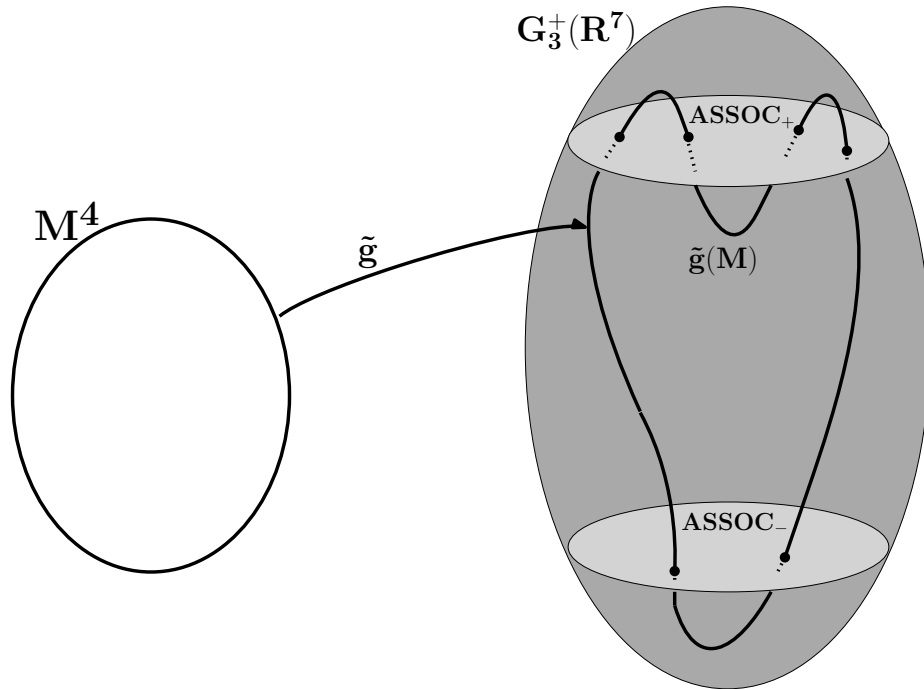


Figure 1: The Gauss map and intersections

Lemma 2.2. *The image $\tilde{g}_*[M] = c [\mathbb{C}P_2] + (c - \chi_M)[\overline{\mathbb{C}P}_2] \in H_4(G_3^+\mathbb{R}^7; \mathbb{R})$ for some $c \in \mathbb{R}$.*

¹See [Fano2] for a discussion on the topological embedding problem.

Proof. Suppose $\tilde{g}_*[M] = c[\mathbf{CP}_2] + d[\overline{\mathbf{CP}}_2]$ for some $c, d \in \mathbb{R}$. We are given the characteristic numbers $e(F)[\mathbf{CP}_2] = 1$ and $e(F)[\overline{\mathbf{CP}}_2] = -1$ by Theorem 2.1 above. Using these we compute

$$\langle e(F_3^7), \tilde{g}_*[M] \rangle = c - d.$$

Also

$$\langle e(F_3^7), \tilde{g}_*[M] \rangle = \langle \tilde{g}^*e(F), [M] \rangle = \langle e(TM), [M] \rangle = \chi_M.$$

Equating these two results and eliminating the variable $d = c - \chi_M$ yields the lemma. \square

Note that this is also true with integer coefficients since the fourth homology of $G_3^+\mathbb{R}^7$ contains no torsion by [AK16]. Next we use this to compute an intersection number.

Lemma 2.3. *The intersection number $\tilde{g}_*[M] \bullet [ASS] = c$.*

Proof. Making use of the above information we compute the following.

$$\begin{aligned} \tilde{g}_*[M] \bullet [ASS] &= (c[\mathbf{CP}_2] + (c - \chi_M)[\overline{\mathbf{CP}}_2]) \bullet [ASS] \\ &= c\langle [ASS], \text{PD}^{-1}[\mathbf{CP}_2] \rangle + (c - \chi_M)\langle [ASS], \text{PD}^{-1}[\overline{\mathbf{CP}}_2] \rangle \\ &= c\langle [ASS], \frac{p_1E + eF}{2}eF \rangle + (c - \chi_M)\langle [ASS], \frac{p_1E - eF}{2}eF \rangle \\ &= (c - \chi_M/2)\langle [ASS], p_1EeF \rangle + \chi_M/2\langle [ASS], e^2F \rangle \\ &= (c - \chi_M/2) \cdot 1 + \chi_M/2\langle [ASS], p_1^2E \rangle \\ &= (c - \chi_M/2) + \chi_M/2 \\ &= c, \end{aligned}$$

where we have used the Theorem 2.1 again. We used part (d) by dualizing the projective spaces for illustration purposes but employing (c) and dualizing associatives again would also suffice. Here $[ASS]$ denotes the associative subspace of the Grassmannian. This is the subset of the Grassmannian which correspond to the 3-planes in \mathbb{R}^7 calibrated by the 3-form ϕ . \square

Rather than pushing the chain forward, we can use pullbacks of forms as well again to recompute the same number.

Lemma 2.4. *Alternatively we can compute the intersection number as*

$$\tilde{g}_*[M] \bullet [ASS] = \frac{1}{2}(\chi - 3\tau).$$

Also we compute

$$\tilde{g}_*[M] \bullet [\widetilde{ASS}] = -\frac{1}{2}(\chi + 3\tau).$$

Proof. We use the Theorem 2.1 so that,

$$\begin{aligned}
\tilde{g}_*[M] \bullet [ASS] &= \langle \text{PD}^{-1}[ASS], \tilde{g}_*[M] \rangle \\
&= \langle \frac{1}{2}(p_1E + eF), \tilde{g}_*[M] \rangle \\
&= \frac{1}{2} \langle p_1(*^*E) + e(*^*F), g_*[M] \rangle \\
&= \frac{1}{2} \langle p_1F + eE, g_*[M] \rangle \\
&= \frac{1}{2} \langle -p_1E + eE, g_*[M] \rangle \\
&= \frac{1}{2} \langle -p_1(TM) + e(TM), [M] \rangle.
\end{aligned}$$

because $E \oplus F = \underline{\mathbb{R}}^7$ is trivial, $g^*(E) = TM$ and the Hirzebruch signature formula [Hir95]. Reversed oriented associative Grassmannian case is similar. \square

Comparing the two Lemmata we obtain the value $c = \frac{1}{2}(\chi - 3\tau)$. This improves our Lemma 2.2 and gives us the full fundamental class formula as follows.

Proposition 2.5. *The image of the canonical class in $H_4(G_3^+ \mathbb{R}^7; \mathbb{R})$ is given by*

$$\tilde{g}_*[M] = \frac{1}{2}(\chi - 3\tau) [\mathbb{C}\mathbb{P}_2] - \frac{1}{2}(\chi + 3\tau) [\overline{\mathbb{C}\mathbb{P}_2}]$$

for any immersion $i : M \rightarrow \mathbb{R}^7$.

In particular, this tells us that the image of the orientation class is independent from the immersion. This result is used in the proof of a main result in [Ü15]. If $\chi = \tau = 0$ then according to our Proposition 2.5 intersection numbers of the image of the canonical class with coassociatives is automatically zero. Now, from this point on assume that the immersed 4-manifold is coassociative-free. We are ready to prove our vanishing result.

Theorem 2.6 (Vanishing). *If M^4 is closed and $i : M \rightarrow \mathbb{R}^7$ is a coassociative($*\phi_0$)-free immersion, then the Euler characteristic χ_M and the signature τ_M vanishes.*

Proof. Since through the star map we have

$$g_*[M] \bullet [COASS] = *_* \circ g_*[M] \bullet *_*[COASS] = \tilde{g}_*[M] \bullet [ASS],$$

the answer c above is the intersection of the image of the tangent planes with the coassociative planes in $G_4^+ \mathbb{R}^7$. Free condition implies that the intersection numbers of tangent planes of M with coassociative and reversed oriented coassociatives are zero. These are the intersection numbers in Lemma 2.4 and gives the

linear system

$$\begin{aligned} 0 &= \frac{1}{2}(\chi - 3\tau) \\ 0 &= -\frac{1}{2}(\chi + 3\tau). \end{aligned}$$

□

This is in accordance with the following generalization of a theorem of [Ü11]. This Theorem tells us that the Euler characteristic is zero even for any manifold with G_2 structure instead of the Euclidean 7-space.

Theorem 2.7 (χ vanishing). *Let $i : M \rightarrow X$ be a coassociative-free immersion of a smooth 4-manifold into a 7-manifold (X, φ) with a G_2 structure. Then the Euler characteristic χ_M vanishes.*

The same proof carries on if one takes the nowhere vanishing three form,

$$\eta := i^*(\varphi|_{i(M)}) \neq 0$$

provided by $*\varphi$ freeness. Then one have to use an arbitrary metric to take its Hodge star $*\eta \in \Lambda^1 M$ and convert to a nowhere vanishing vector field on M by the metric duality.

3 Cayley-free Embeddings

The intersection theoretic computation techniques that we used in the previous section can also be used in the Cayley case as well. This case is easier because the fundamental class formula is known already. Let $f : M \rightarrow \mathbb{R}^8$ be an immersion of a compact oriented 4-manifold, then its Gauss map $g : M \rightarrow G_4^+ \mathbb{R}^8$ is computed as [SZ14],

$$g_*[M] = \frac{1}{2}\chi[G(4,5)] + \lambda[G(1,5)] + \frac{3}{2}\tau[G(2,4)].$$

Here $\lambda = \frac{1}{2}g^*eF[M]$ and $\tau = \tau(M) = \frac{1}{3}g^*p_1E[M] = \frac{1}{3}p_1[TM]$ is the signature. We will be dealing with the embedding case, so that $\lambda = 0$. We intersect the Gauss image of the embedded 4-manifold with the Cayley and anti-Cayley planes. We can compute the first intersection number as follows,

$$\begin{aligned} g_*[M] \bullet [CAY] &= \langle PD^{-1}[CAY], g_*[M] \rangle \\ &= \langle \frac{1}{2}(p_1E - eE + eF), \frac{\chi}{2}[G(4,5)] + \frac{3\tau}{2}[G(2,4)] \rangle \\ &= -\frac{1}{2}(\chi - 3\tau) \end{aligned}$$

using the integration table in page 517. To be able to compute the intersection number with the negative Cayley locus, we have to figure out the Poincaré dual. Reversing the orientation of the Cayley planes, orientation of the bundles are reversed, so that the sign of the Euler classes are changed however the first Pontrjagin class does not change sign. So that the Poincaré dual becomes,

$$\text{PD}^{-1}[\widetilde{\text{CAY}}] = \frac{1}{2}(p_1E + eE - eF).$$

Inserting this, we compute the second intersection number as follows,

$$\begin{aligned} g_*[M] \bullet [\widetilde{\text{CAY}}] &= \langle \text{PD}^{-1}[\widetilde{\text{CAY}}], g_*[M] \rangle \\ &= \frac{1}{2}(\chi + 3\tau). \end{aligned}$$

Cayley-free condition implies that these two intersection numbers has to be zero, so the linear equations implies the vanishing similarly as in the previous section.

Theorem 3.1 (Vanishing). *If M^4 is compact and $i : M \rightarrow \mathbb{R}^8$ is a Cayley(ψ_0)-free embedding, then the Euler characteristic χ_M and the signature τ_M vanishes.*

Using the appropriate h-principle technique, this result can be extended to the Cayley free embeddings into 8-manifolds with $Spin_7$ structure. See [Ü18] for details.

4 Contractibility of the Gauss map

In this section we will focus on the Gauss map, and prove parallelizability through it. We shall start with proving that the Hurewicz homomorphism

$$h_n : \pi_n(G_3^+ \mathbb{R}^7) \longrightarrow H_n(G_3^+ \mathbb{R}^7; \mathbb{Z})$$

is injective at the level $n = 4$. This homomorphism is defined by sending a homotopy class to its homology class. So that $h_n([f]) = f_*[S^n]$ where f represents a homotopy class, and f_* is the push forward at the homology level.

The idea of the proof is to use the generalized *Mod- \mathcal{C}_p Hurewicz Theorems*. We start with an introduction to them which follows [DK01]. Alternative classical resources are [MT68] and [Hu59]. For a subset P of prime numbers, let \mathcal{C}_P denote the class of torsion abelian groups which has no elements of order a positive power of $p \in P$. This class actually satisfies the properties of a so-called *Serre Class*. If we take $P = \{p\}$ then we just use the notation \mathcal{C}_p . So as an example we have

$$\mathcal{C}_7 = \text{Torsion abelian groups which has no element of order } 7^k, k \in \mathbb{Z}^+.$$

Obviously the groups $\mathbb{Z}_5, \mathbb{Z}_{24}$ etc. but \mathbb{Z}_{49} are in this category. A homomorphism $\varphi : A \rightarrow B$ between two abelian groups is called a \mathcal{C}_p -monomorphism if $\text{Ker } \varphi \in \mathcal{C}_p$, a \mathcal{C}_p -epimorphism if $\text{Coker } \varphi \in \mathcal{C}_p$ and a \mathcal{C}_p -isomorphism if both of these conditions are satisfied. Now we are ready to state the following classical theorem whose proof involves spectral sequences [DK01].

Theorem 4.1 (Mod- \mathcal{C}_p Hurewicz). *Let X be 1-connected and $\pi_i(X) \in \mathcal{C}_p$ for all $i < n$. Then, $H_i(X; \mathbb{Z}) \in \mathcal{C}_p$ for all $0 < i < n$ and the Hurewicz map $h_n : \pi_n(X) \rightarrow H_n(X; \mathbb{Z})$ is a \mathcal{C}_p -isomorphism.*

As an application we obtain a central result of this section.

Lemma 4.2 (Injectivity). *The Hurewicz homomorphism*

$$h_4 : \pi_4(G_3^+ \mathbb{R}^7) \longrightarrow H_4(G_3^+ \mathbb{R}^7; \mathbb{Z})$$

is injective.

Proof. Following [AK16], $G_3^+ \mathbb{R}^7$ is simply-connected so 1-connected, and possessing $\pi_{0123} = \{0, 0, \mathbb{Z}_2, 0\}$ as the first four homotopy groups none of which contains elements of order 7^k , $k \in \mathbb{Z}^+$ hence of class \mathcal{C}_7 .

Taking $X = G_3^+ \mathbb{R}^7$ and $n = 4$ in the above Mod- \mathcal{C}_7 Hurewicz Theorem 4.1, we get that the Hurewicz map $h_4 : \pi_4(X) \rightarrow H_4(X; \mathbb{Z})$ is a \mathcal{C}_7 -isomorphism. In particular $\text{Ker } h_4 \in \mathcal{C}_7$. Besides that $\text{Ker } h_4$ is certainly contained in $\pi_4(G_3^+ \mathbb{R}^7) = \mathbb{Z} \oplus \mathbb{Z}$ from [AK16] which is torsion free. \mathcal{C}_7 is a class of torsion groups, and the only torsion subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ is the trivial one, hence $\text{Ker } h_4 = 0$. \square

It is a curious question whether this map is surjective as well. We leave it as an exercise to the reader. Covering all homology classes by means of spheres does not seem likely in a middle level though.

Combining the Proposition 2.5 and the vanishing Theorem 2.6 we obtain the following consequence.

Corollary 4.3. *If M^4 is compact and $i : M \rightarrow \mathbb{R}^7$ is a coassociative($\ast\phi_0$)-free immersion, then $g_*[M] = 0$ in $H_4(G_4^+ \mathbb{R}^7; \mathbb{Z})$ for the associated Gauss map g .*

This observation eventually leads to the contractibility of the Gauss map.

Theorem 4.4. *If M^4 is a compact, oriented, spin manifold and $i : M \rightarrow \mathbb{R}^7$ is a coassociative ($\ast\phi_0$) free immersion, then its image $g(M)$ under the Gauss map $g : M \rightarrow G_4^+ \mathbb{R}^7$ is contractible, so that M is parallelizable.*

Proof. The proof uses the obstruction theory. See [Hu59, MS74]. We will work with the equivalent map $\tilde{g} : M \rightarrow G_3^+ \mathbb{R}^7$ for convenience. We shrink the map skeleton by skeleton. Restriction of g to the 0-th and 1-st skeleton of M is easily contracted to a point by a homotopy because the underlying space $G_4^+ \mathbb{R}^7$ is connected and simply connected. Next comes the problem of shrinking the map over the 2-skeleton.

$$\tilde{g} : M_{(2)} / M_{(1)} \rightarrow G_3^+ \mathbb{R}^7.$$

Since we already shrunk the map over the 1-skeleton, this gives a cocycle hence a cohomology class in the second cohomology of M with π_2 coefficients.

$$\mathfrak{o}_2 \in H^2(M; \{\pi_2(G_3^+ \mathbb{R}^7)\}) = H^2(M; \mathbb{Z}_2).$$

The Stiefel-Whitney class w_2 is equal to the obstruction \mathfrak{o}_2 . Since M is spin, by our assumption we get rid of this obstruction and the next one is

$$\mathfrak{o}_3 \in H^3(M; \{\pi_3(G_3^+ \mathbb{R}^7)\}) = 0,$$

since the homotopy group $\pi_3(G_3^+ \mathbb{R}^7)$ is trivial as computed in [AK16]. The last obstruction lies

$$\mathfrak{o}_4 \in H^4(M; \{\pi_4(G_3^+ \mathbb{R}^7)\}) = H^4(M; \{\mathbb{Z} \oplus \mathbb{Z}\}).$$

Since M is a smooth 4-manifold and the top homology is either \mathbb{Z} or trivial, we can use a cell complex decomposition with only one 4-cell. After we have contracted the 3-skeleton of M to a point in $G_3^+ \mathbb{R}^7$, the 4-cell gives a map $S^4 \rightarrow G_3^+ \mathbb{R}^7$, homology class of which is the same as $\tilde{g}_*(M)$. We have already shown that the homology class $[\tilde{g}_*(M)]$ is trivial in $H_4(G_3^+ \mathbb{R}^7; \mathbb{Z})$. In Section 4 we have shown that the Hurewicz homomorphism

$$h_4 : \pi_4(G_3^+ \mathbb{R}^7) \rightarrow H_4(G_3^+ \mathbb{R}^7; \mathbb{Z})$$

is injective. Hence the homotopy class of the 4-cell map is trivial, consequently $\tilde{g}(M)$ is contractible. Hence M has trivial tangent bundle. \square

5 Some examples

In this section we will give some examples to illustrate our theorems. In particular when we lift some assumptions we will see that there are spaces which become no longer embeddable in the appropriate way. Our main theorem applies to some connected sums of the panelled web 4-manifolds denoted by $M_n^1, M_{g,n}^2, M_{g,n}^3, M_n^4$ which are constructed in [AK12] and also $M_{g,n}^5$ is defined in [KOA13]. These manifolds are constructed using some special type of Kleinian groups that goes under the same name. They actually come up with a Riemannian metric which is locally conformally flat. But we are only interested in their underlying smooth structure here. Among their topological invariants, their Euler characteristics are computed as follows.

$$\chi(M_n^1) = -4g, \quad \chi(M_{g,n}^2) = \chi(M_{g,n}^3) = \chi(M_{g,n}^5) = 4 - 4g - 4n, \quad \chi(M_n^4) = -2n.$$

Since these spaces come up with locally conformally flat metrics their signature is zero. The following examples satisfy all the hypothesis of our Theorem 4.4 and Theorem 2.7. However their tangent bundle is non-trivial because of the

non-triviality of the signature obstruction $p_1[M] = 3\tau(M) \neq 0$. In the following Corollary, $K3$ denotes the underlying smooth manifold of a smooth quartic in complex projective space, namely the $K3$ Surface.

Corollary 5.1. *The following families of 4-manifolds*

1. $M_{11k}^1 \# 2kK3$ for all $k > 0$
2. $M_{g,11k-g}^{2,3,5} \# 2kK3$ for all $11k > g > 0$
3. $M_{11k-2}^4 \# kK3$ for all $k > 0$

have the invariants $w_{1234} = 0$, $\chi = 0$ but are not parallelizable, so they do not admit any coassociative-free immersions into \mathbb{R}^7 , as well as Cayley-free immersions into \mathbb{R}^8 .

Proof. We will check the invariants of the 4-manifolds. Let us start from the Euler characteristic and signature. Using the connected sum formula for the Euler characteristic we obtain the following.

$$\chi(M_{11k}^1 \# 2kK3) = \chi(M_{11k}^1) - 2k\chi(S^4) + 2k\chi(K3) = -4 \cdot 11k - 2k \cdot 2 + 2k \cdot 24 = 0.$$

Orientability and being spin is preserved under the connected sum operation. Since all the building blocks we use are orientable and spin, so their connected sum hence the first and second Stiefel-Whitney classes vanish: $w_1 = 0$ and $w_2 = 0$ as obstructions. The first Steenrod operator coincides with the Bockstein homomorphism [MT68] and applying the Wu's explicit formula [MS74] and orientability we get

$$\beta w_2 = Sq^1 w_2 = w_1 w_2 + w_3 = w_3.$$

See also [Hat18]. Because this is a homomorphism, $w_3 = 0$ as well. We have already computed the Euler characteristic as zero, so its mod 2 reduction, hence $w_4 = 0$. The signature is additive under the connected sum operation, so,

$$\tau(M_{11k}^1 \# 2kK3) = \tau(M_{11k}^1) + 2k\tau(K3) = 0 + 2k \cdot (-16) = -32k,$$

which is nontrivial, so that the tangent bundle is nontrivial as well. □

Consequently these examples show that the vanishing of signature is a crucial necessary condition for coassociative free and Cayley free immersions/embeddings of 4-manifolds into manifolds with G_2 and $Spin_7$ -structure, respectively. The parallel web manifolds with even indices have vanishing signature and other invariants. On the other hand, they have strictly negative Euler characteristic, hence they do not embed in a free way to any manifold with G_2 or $Spin_7$ structure. Other family of examples can be constructed using surgeries, like infinite family of homotopy $K3$ surfaces, for example knot surgered symplectic homotopy $K3$ surfaces $E(2)_K$, where K is any fibered knot. See [CK11] and references therein for an overview and current results in the subject.

We also give examples of parallelizable 4-manifolds with complicated non-simply-connected topology. Here $S^2 \times S^2$ stands for the 4-manifold which is the product of two 2-spheres.

Theorem 5.2. *The following families of 4-manifolds are parallelizable.*

1. $M_g^1 \# 2gS^2 \times S^2$ for all $g > 0$
2. $M_{g,n}^{2,3,5} \# (2g + 2n - 3)S^2 \times S^2$ for all $g, n > 0$
3. $M_n^4 \# (n - 1)S^2 \times S^2$ for all $n > 0$.

Proof. Again we are supposed to check the invariants. Taking into consideration that $\chi(S^2 \times S^2) = 4$ and $\tau(S^2 \times S^2) = 0$ the proof is similar to that of the Corollary 5.1. One can check their invariants as $w_{1234} = e = p_1 = 0$. So by the classical parallelizability Theorem 1.2 these 4-manifolds all have trivial tangent bundle. \square

Consequently they satisfy the necessary conditions of our vanishing Theorems 2.6 and 3.1. Then combining with the results in [Ü15] and [Ü18] we can conclude that they have coassociative-free and Cayley-free embeddings into \mathbb{R}^7 or \mathbb{R}^8 and other manifolds with G_2 structure or $Spin_7$ structure which are flat in a neighborhood of a point, respectively. Hence these 4-manifolds with arbitrarily large fundamental groups (also arbitrarily large first or second Betti number) are freely embeddable into flat tori like \mathbb{T}^7 or \mathbb{T}^8 .

ORTA MH. ZÜBEYDE HANIM CD. NO 5-3 MERKEZ 74100 BARTIN , TÜRKİYE.
E-mail address: kalafat@math.msu.edu

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