On the Curvature of Einstein-Hermitian Surfaces

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April 8, 2019

Abstract

We give a mathematical exposition of the Page metric, and introduce an efficient coordinate system for it. We carefully examine the submanifolds of the underlying smooth manifold, and show that the Page metric does not have positive holomorphic bisectional curvature. We exhibit a holomorphic subsurface with flat normal bundle. We also give another proof of the fact that a compact complex surface together with an Einstein-Hermitian metric of positive orthogonal bisectional curvature is biholomorphically isometric to the complex projective plane with its Fubini-Study metric up to rescaling. This result relaxes the Kähler condition in Berger's theorem, and the positivity condition on sectional curvature in a theorem proved by the second author.

1 Introduction

Let (M, J) be a complex manifold. A Riemannian metric g on M is called *Hermitian* if the complex structure $J : TM \to TM$ is an orthogonal transformation at every point on M with respect to the metric g, that is, $g(X,Y) = g(JX, JY)$ for tangent vectors $X, Y \in T_pM$ for all $p \in M$. In this case, the triple (M, g, J) is called a *Hermitian manifold*. For Hermitian metrics we have further notions of curvature related to complex structure: The *holomorphic bisectional curvature* in the direction of a pair of unit tangent vectors $X, Y \in T_pM$ is defined as

$$\mathrm{H}(X,Y) := \mathrm{Rm}(X, JX, Y, JY).$$

If one applies the algebraic Bianchi identity, and J-invariance in the Kähler case, it is easy to see that this is the sum of sectional curvatures of the planes spanned by X, Y and X, JY. We call this as the *summation identity* in the Kähler case which gives some visual insight. Bisectional curvature is actually *not* an invariant of the plane spanned by the vectors X, Y. Rather, it is an invariant of the holomorphic planes spanned by them. As a special case, if one takes the two vectors identical, then the result coincides with the sectional curvature of the holomorphic plane spanned. This is called the *holomorphic (sectional) curvature* in that direction vector. Although it can be considered as a map on the sphere $S^{2n-1}(T_pM) \to \mathbb{R}$ at each point, just in the case of bisectional curvature, this map is an invariant of the holomorphic plane, and therefore can be considered as a map on the complex Grassmannian of one lower real dimension since it is constant on the Hopf circle fibers. Positivity of the sectional curvature implies that of bisectional curvature, which implies positivity of holomorphic curvature. However, the converses are not necessarily true in general.

In this paper, we work on some explicit 4-manifolds to understand various notions of curvature. We also prove a uniformization theorem for positive bisectional curvature. First, let us review some well-known theorems in special cases.

Theorem 1.1 (Frankel conjecture, Siu-Yau Thm [SY80]). *Every compact Kähler manifold of positive bisectional holomorphic curvature is biholomorphic to the complex projective space.*

This theorem does not, however, specify the metric in question. Nevertheless, if we in addition assume that the metric is *Einstein*, then the metric is unique, too:

Theorem 1.2 ([Ber65, GK67]). *An n-dimensional compact connected Kähler manifold with an Einstein (or constant scalar curvature) metric of positive holomorphic bisectional curvature is globally isometric to $\mathbb{CP}_n$ with the Fubini-Study metric (up to rescaling).*

Our aim here is to relax the Kähler condition on the metric in Theorem 1.2 to merely being Hermitian in dimension 4. In this case the summation identity is no longer valid. Our main theorem is:

Theorem 1.3. *If (M, g, J) is a compact complex surface together with an Einstein-Hermitian metric of positive holomorphic bisectional curvature, then it is biholomorphically isometric to $(\mathbb{CP}_2, g_{FS})$, the complex projective plane with its Fubini-Study metric up to rescaling.*

We note that an analogous result with the positivity assumption on the *sectional* curvature is proved in [Koc14] by the second author. However, in the Hermitian case, positivity of sectional and bisectional curvatures are not related in general due to invalidity of the summation identity. Consequently the techniques in that paper cannot be directly applied here.

Einstein-Hermitian metrics on compact complex surfaces are classified by LeBrun:

Theorem 1.4 ([LeB12]). *If (M, g, J) is a compact complex surface together with an Einstein-Hermitian metric, then only one of the following holds.*

1. *g is Kähler-Einstein(KE).*
2. *M is biholomorphic to $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ and g is the Page metric (up to rescaling).*
3. *M is biholomorphic to $\mathbb{CP}_2 \sharp 2\overline{\mathbb{CP}}_2$ and g is the Chen-LeBrun-Weber metric (up to rescaling).*

In other words, an Einstein-Hermitian metric is either Kähler-Einstein to start with, or is one of the two exceptional models. These exceptional Einstein metrics are non-Kähler, but they are *conformally Kähler*. By Theorem 1.2 we only need to consider the non-Kähler case. We start with the Page metric case. We give an introduction to this metric, and show that it does not have positive curvature everywhere. One of the key facts in the proof of this result is Frankel's Theorem [Fra61] which states that totally geodesic submanifolds of complementary dimensions on positively curved manifolds necessarily intersect. Since the Page metric has an explicit form, we are also able to give a computational proof of the failure of positivity. Secondly we introduce Euler coordinates, and use Dragomir-Grimaldi's theorem [DG91] to get the analogous result on bisectional curvature. These coordinates are especially useful for Page metric and its submanifolds. We hope that they will be useful for others who are interested in local computations in 4-dimensional geometry. For another application of these coordinates see [KS16].

On the other hand, the Chen-LeBrun-Weber metric does not have such an explicit formula. Therefore, this explicit analysis is not an available option at this time. We need to prove a more general result to handle this case. Using curvature estimates and Weitzenböck formula techniques we prove the following result.

Theorem 4.4. *Let M be a compact Einstein-Hermitian 4-manifold of positive holomorphic bisectional curvature. Then the Betti number b^2_- vanishes.*

As a consequence we have,

Corollary 1.5. *There is no Einstein-Hermitian metric of positive holomorphic bisectional curvature on any blow up of $\mathbb{CP}_2$.*

Since the underlying smooth 4-manifolds of the exceptional cases $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2 \sharp 2\overline{\mathbb{CP}}_2$ have non-zero b^2_-, we deduce the following corollary.

Corollary 1.6. *The Page and Chen-LeBrun-Weber metrics are not of positive holomorphic bisectional curvature.*

This eliminates the later two possibilities in LeBrun's Classification Theorem 1.4. In the remaining Kähler-Einstein case, we apply the Berger – Goldberg-Kobayashi Theorem 1.2, and the proof of our Theorem 1.3 follows.

We note that the authors proved a more general result in a different article [KK15] in the conformally Kähler case, which uses slightly different Weitzenböck techniques. Besides that, the discussion on Page metric and Euler coordinate computations are the main and completely new material here. We give a better understanding of this important metric to the reader in this paper.

In §2 we give a careful topological analysis of the Page metric and show that it does not have positive sectional curvature everywhere. In §3 we provide Euler coordinates and give an alternative proof of the fact that it the Page metric is not of positive holomorphic bisectional curvature by exhibiting a subsurface with flat normal bundle. In §4 we prove some estimates and classify Einstein-Hermitian 4-manifolds of positive bisectional curvature.

Acknowledgements. We thank Claude LeBrun for his encouragement. Also thanks to Ö. Kelekçi and C. Tezer for useful discussions. This work is partially supported by the grant ♯113F159 of Tübitak (Turkish science and research council).

2 Page metric

In this section we give a rigorous mathematical exposition of the Page metric, and describe its topology in detail. At the end we prove that it does not have positive sectional curvature everywhere. This section and its figures are part of C. Koca's thesis [Koc12].

The Page metric was discovered by D. Page in 1978 as a limiting metric of Kerr-de Sitter solution (see [Pag78]). It is the unique Einstein-Hermitian non Kähler metric on the blow up of complex projective plane. To define it formally, we first think of the following metric on the product $S^3 \times I$ where I is the closed interval $[0, \pi]$:

$$g = V(r)dr^2 + f(r)(\sigma_1^2 + \sigma_2^2) + \frac{C\sin^2 r}{V(r)}\sigma_3^2$$

where the coefficient functions are given by the following expressions [1]

$$V(r) = \frac{1 - a^2\cos^2 r}{3 - a^2 - a^2(1+a^2)\cos^2 r}$$

$$f(r) = 4\,\frac{1 - a^2 cos^2 r}{3 + 6a^2 - a^4}$$

$$C = \frac{1}{(3+a^2)^2}$$

[1] In the original paper [Pag78] of Page, there is a typo in the equation (41). The last term inside the curly brackets has to be divided by 4 and $(3-\nu^2)^2$ replaced by $(3+\nu^2)^2$. See [Pag09] for the information.

and a is the unique positive root of $a^4 + 4a^3 - 6a^2 + 12a - 3 = 0$. Here, $\sigma_1, \sigma_2, \sigma_3$ is the standard left invariant 1-forms on the Lie group $SU(2) \approx S^3$. At the endpoints $r = 0$ and π, we see from the formula that the metric shrinks to a round metric on S^2. Thus, g descends to a metric, denoted by g_{Page}, on the quotient $(S^3 \times I)/\sim$ where $\sim$ identifies the fibers of the Hopf fibration $p : S^3 \to S^2$ on the two ends $S^3 \times \{0\}$ and $S^3 \times \{\pi\}$ of the cylinder $S^3 \times I$. See Figure 1. The resulting

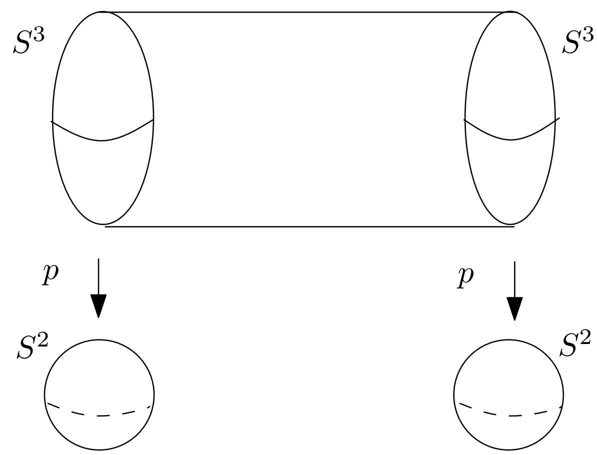

Figure 1: *The manifold $S^3 \times I$ with its two boundary components.*

manifold is indeed the connected sum $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$. To see this, recall that in the cell decomposition of $\mathbb{CP}_2$, the attaching map from the boundary of the 4-cell (which is S^3) to the 2-skeleton (which is $\mathbb{CP}_1 \approx S^2$) is given by the Hopf map [Hat02]. So, if we cut the cylinder $S^3 \times I$ in two halves and identify the Hopf fibers of S^3 at each end, we get $\mathbb{CP}_2 - \{\text{small ball}\}$. Since the right and left halves have different orientations, we obtain $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ in the quotient. See Figure 2 for assistance. Next, we will prove that the Page metric is not of positive sectional curvature. We will use the following classical theorem by Frankel.

Theorem 2.1 ([Fra61]). *Let M be a smooth n-manifold, and let g be a complete Riemannian metric of positive sectional curvature. If X and Y are two compact totally geodesic submanifolds of dimensions d_1 and d_2 such that $d_1 + d_2 \geq n$, then X and Y intersect.*

In our case, the two 2-spheres on each end of the above quotient will play the role of X and Y. They are compact and the dimensions add up to 4. So it remains to show that those two submanifolds are totally geodesic with respect to g_{Page}. Since they are obviously disjoint, this will imply that g_{Page} cannot have positive sectional curvature. There is a very well-known lemma to detect totally geodesic submanifolds:

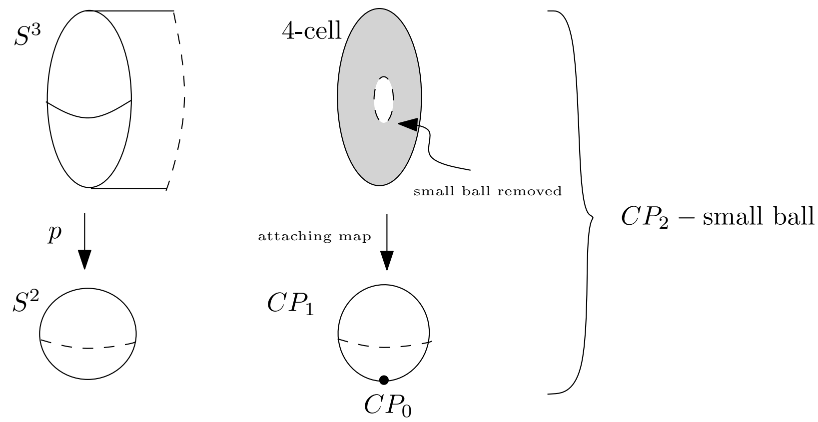

Figure 2: *Boundary identifications to produce the closed 4-manifold* $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$.

Lemma 2.2. *Let (M, g) be a Riemannian manifold. If f is an isometry, then each connected component of the fix point set* $\mathrm{Fix}(f)$ *of f is a totally geodesic submanifold of M.*

So, below we will show that there is an isometry of the Page metric whose fix point set is precisely the two end spheres. What are the isometries of the Page metric? Derdziński showed that the Page metric is indeed conformal to one of Calabi's extremal Kähler metrics on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$ in [Der83]. On the other hand, the identity component of the isometry group of extremal Kähler metrics is a maximal compact subgroup of the identity component of the automorphism group [Cal85]. In the case of $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$, this implies that the identity component of the isometry group of the Page metric is $U(2) = (SU(2) \times S^1)/\mathbb{Z}_2$. By the formula of the metric, we see that the isometries in the $SU(2)$ component are precisely given by the left multiplication action of $SU(2)$ on the first factor of $S^3 \times I$. Note that the forms σ_i, $i = 1, 2, 3$ are invariant under the action, but the action on the 3-spheres $S^3 \times \{r\}$, $r \in (0, \pi)$ is fixed-point- free! The metric is invariant under this action as the coefficients of the metric only depend on the parameter r.

Now, let us see what happens at the endpoints $r = 0$ and $r = \pi$: It is well-known that the action of $U \in SU(2)$ on the 2-sphere S^2 (after the quotient) is given by the conjugation $A \mapsto UAU^{-1}$, where we regard the 2×2 complex matrix $A = x\sigma_1 + y\sigma_2 + z\sigma_3$ with $x^2 + y^2 + z^2 = 1$ as a point of S^2. It is now straightforward to see that the action of $-I \in SU(2)$ is trivial on S^2 (since $(-I)A(-I)^{-1} = A$); thus, it fixes every point on S^2. Therefore, we conclude that the fixed point set of the isometry given by the "antipodal map" $-I \in SU(2)$ consists of the two 2-spheres at each end of the quotient $\big((S^3 \times I)/\sim\big) \approx \mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. Note that, indeed, there is an S^1-family of isometries generated by rotation in direction of σ_3 having the exact same fixed point set. So we showed that there are two disjoint compact totally geodesic submanifolds of $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. Therefore, Frankel's theorem implies the following.

Theorem 2.3. *The sectional curvature of the Page metric is not everywhere positive.*

Finally, we note that we can actually show the failure of positivity *directly by brute-force* using tensor calculus: Introduce a new coordinate function $x := \cos(r)$, so that the metric becomes

$$g = U^2(x)dx^2 + g^2(x)(\sigma_1^2 + \sigma_2^2) + \frac{D^2}{W(x)}\sigma_3^2$$

where the coefficient functions are given as

$$U(x) = \sqrt{\frac{1 - a^2x^2}{(3 - a^2 - a^2(1 + a^2)x^2)(1 - x^2)}}$$

$$g(x) = 2\sqrt{\frac{1 - a^2x^2}{3 + 6a^2 - a^4}}$$

$$D = \frac{1}{3 + a^2}$$

and choose the following vierbein: $\{e^0, e^1, e^2, e^3\} := \{Udx, g\sigma_1, g\sigma_2, DU^{-1}\sigma_3\}$. Then by a standard tensor calculus, we see that the sectional curvature of the plane generated by e_0 and e_1 is given by

$$K_{01} = 2\,\frac{g'U' - g''U}{gU^3}.$$

Using a computer program like *Maple*, one can easily verify that this function $K_{01}(x)$ can take both positive and negative values for $x \in (-1, 1)$.

3 Euler coordinates and Flat bundles

In this section we will introduce an efficient coordinate system and use it to show that the Page metric is not of positive holomorphic bisectional curvature by explicitly analyzing the submanifolds of the underlying smooth manifold. We would like to use the following theorem of Dragomir and Grimaldi. See the book [DO98] on locally conformal Kähler (l.c.K.) geometry p.157 for an exposition.

Theorem 3.1 ([DG91])**.** *Let S be a complex submanifold of the l.c.K. manifold M. If M has positive holomorphic bisectional curvature everywhere, then the normal bundle of the given immersion $S \subset M$ admits no parallel sections.*

In order to make use of this theorem, we need to analyze the complex submanifolds of $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$. For this purpose we use *Euler angles* [Zhao4] on the $S^3 \subset \mathbb{R}^4$ which e.g. realizes the Hopf fibration in the best.

$$0 \leq \theta \leq \pi, \;\; 0 \leq \phi \leq 2\pi, \;\; 0 \leq \psi \leq 4\pi$$

$$\begin{aligned}
x_1 &:= r\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2} \\
x_2 &:= \quad '' \quad \sin\frac{\psi+\phi}{2} \\
x_3 &:= r\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2} \\
x_0 &:= \quad '' \quad \sin\frac{\psi-\phi}{2}
\end{aligned}$$

where the *Hopf fibration* in these coordinates is just a projection [FIP04],

$$h : S^3 \longrightarrow S^2, \qquad h(\theta,\psi,\phi) = (-\phi,\theta).$$

Here the exchange $\phi \longleftrightarrow \theta$ is needed to relate to the calculus angles on S^2. Changing ψ does not change the element in the image. So, whenever the image ϕ,θ is fixed, ψ parametrizes the Hopf circle (fiber).

An orthonormal, invariant coframe $\{\sigma_1,\sigma_2,\sigma_3\}$ on S^3 is given as follows:

$$\sigma_1 = (x_1dx_0 - x_0dx_1 + x_2dx_3 - x_3dx_2)/r^2 = (\sin\psi d\theta - \sin\theta\cos\psi d\phi)/2$$

$$\sigma_2 = (x_2dx_0 - x_0dx_2 + x_3dx_1 - x_1dx_3)/r^2 = (-\cos\psi d\theta - \sin\theta\sin\psi d\phi)/2$$

$$\sigma_3 = (x_3dx_0 - x_0dx_3 + x_1dx_2 - x_2dx_1)/r^2 = (d\psi + \cos\theta d\phi)/2.$$

One can check in a straightforward manner the identities,

$$d\sigma_1 = 2\sigma_2 \wedge \sigma_3 \quad \text{and} \quad \sigma_1^2 + \sigma_2^2 = (d\theta^2 + \sin^2\theta d\phi^2)/4.$$

Plugging these into the Page metric's expression we get

$$g_{\text{Page}} = Vdr^2 + \left\{\frac{f}{4}\sin^2\theta + \frac{C\sin^2 r\cos^2\theta}{4V(r)}\right\}d\phi^2 + \frac{C\sin^2 r}{4V(r)}d\psi^2 + \frac{C\sin^2 r\cos\theta}{4V(r)}(d\psi\otimes d\phi + d\phi\otimes d\psi) + \frac{f}{4}d\theta^2.$$

Letting $U := \sqrt{V(r)}$, $h := \sqrt{f}$, $D := \sqrt{C}$ we have the Vierbein i.e. orthonormal coframe

$$\{e^0, e^1, e^2, e^3\} = \left\{ U dr,\ \frac{Dh}{2\sqrt{C\cot^2\theta + Vf\csc^2 r}}\, d\psi, \right.$$

$$\left. \frac{D\sin r}{2U\sqrt{1 + C^{-1}Vf\tan^2\theta\sin^{-2} r}}\,(d\psi + (C^{-1}Vf\tan\theta\sin\theta\sin^{-2} r + \cos\theta)\, d\phi),\ \frac{h}{2}\, d\theta \right\}$$

Now, we are in a position to analyze some of the subsurfaces easily [SM99, And11]. For example keeping r_0, θ_0 fixed and varying ψ, ϕ, one obtains tori. See more on the subsurfaces of the Page space at [KS16]. We are interested in complex submanifolds. For this purpose, this time keep ϕ_0, θ_0 fixed, vary r, ψ to obtain complex spheres i.e. rational curves as follows. This captures a series of nearby Hopf fibers corresponding to $r \in [0, \pi]$ hence a cylinder $S^1 \times I$ inside $S^3 \times I$ which projects to a sphere under Hopf identification of circles at the two ends. These are complex submanifolds, since they correspond to fibers coming from Hirzebruch Surface description/fibration. This follows easily by recalling that the complex line bundle $\mathcal{O}(1)$ over $\mathbb{CP}_1$ can also be described as the quotient

$$(S^3 \times \mathbb{C})/ \propto \quad \text{where} \quad (x, z) \propto (\lambda x, \lambda z), \quad \lambda \in S^1 \subset \mathbb{C}.$$

There is an obvious diffeomorphism between

$$\frac{(S^3 \times [0, \pi))}{\sim} \approx \frac{(S^3 \times [0, \infty))}{\sim} \quad \text{and} \quad \mathcal{O}(1) = (S^3 \times \mathbb{C})/ \propto;$$

and under this map the quotient of the cylinder in the previous paragraph is mapped to the complex fiber of the complex line bundle $\mathcal{O}(1)$ over the point $(\phi_0, \theta_0) \in \mathbb{CP}_1$. Adding the remaining point at infinity shows that the quotient of the cylinder in $(S^3 \times I)/\sim$ corresponds to the Hirzebruch fiber of the first Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{CP}_1$. These fibers are obvously complex rational curves. The Hopf spheres at the two ends of $S^3 \times I$ correspond to $0, \infty$-sections of this fibration

$$\begin{array}{ccc} \mathbb{CP}_1 & \to & \mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \\ & & \downarrow \\ & & \mathbb{CP}_1 \end{array}$$

Now let's find the curvature of the normal bundle of these fibers. Finding nontrivial connection 1-forms of the Page metric with respect to this basis is not an easy task. We refer the reader to [KS16] for this type of approach with a different choice of vierbein. Instead we will compute the Christoffel symbols and the coefficients of the Riemann curvature tensor to figure out the curvature 2-form of the normal plane bundle of the spheres that we are working on. Here the submanifold directions are $0, 1$ and normal bundle directions are $2, 3$. Since $\theta = \theta_0$ is constant (together with ϕ_0), using one of the computer algebra systems the curvature 2-form of the normal bundle $N\mathbb{CP}_1$ can be computed as [EGH80],

$$\begin{aligned}
\tilde{R}^2{}_3 &= d\tilde{\omega}^2{}_3 + \tilde{\omega}^2{}_1 \wedge \tilde{\omega}^1{}_3 \\
&= \tfrac{1}{2}\tilde{R}^2{}_{3cd}\, e^c \wedge e^d \\
&= \frac{D(Vf' + f(V' - 2V\cot r))\sin r \tan\theta}{2f^{3/2}V^{3/2}\sqrt{1 + C^{-1}Vf\csc^2 r \tan^2\theta}}\, dr \wedge d\psi.
\end{aligned}$$

Here recall that V and f are functions of r. If we focus on one of the spheres where $\theta = 0$ and $\phi = \phi_0$, this curvature 2-form vanishes. So that the normal bundle of this type of sphere is flat. Then the parallel translation on this sphere depends only on the homotopy class, which is unique because of simple-connectivity. Hence, parallel translation is totally path independent. Now starting with two linearly independent vectors of the normal bundle at a point, one extends them by parallel translation to the whole sphere. Since lengths are preserved during the process, these extensions are nowhere vanishing. So we obtain two parallel sections of the normal bundle. Two nowhere zero sections of a plane bundle trivializes it, so we have a trivial, flat normal bundle. We are ready to state the main result of this section.

Theorem 3.2. *The holomorphic bisectional curvature of the Page metric on $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ is not everywhere positive.*

Proof. The spheres above are complex submanifolds and the Page metric is conformally Kähler so certainly l.c.K. Their normal bundle has nontrivial parallel sections. Therefore, we can apply Theorem 3.1. □

4 Estimates and the bisectional curvature

We first describe the 2-form interpretation of the planes in the tangent space which will be very useful. Sectional curvatures at a point $p \in M$ can be thought as a function on the Grassmannian of oriented two planes in the corresponding tangent space.

$$\sec : G_2^+(T_pM) \longrightarrow \mathbb{R}$$

In dimension 4 we have a nice description of this Grassmannian in terms of forms

$$G_2^+(\mathbb{R}^4) \approx \{(\alpha, \beta) \in \Lambda^2_+ \oplus \Lambda^2_- : |\alpha| = |\beta| = 1/\sqrt{2}\} \approx S^2 \times S^2.$$

See [Via11] as a reference. Starting with a plane π, one can choose a special orthogonal basis $\{e_1, e_2\}$ which corresponds to the form $\sigma = e^1 \wedge e^2 \in \Lambda^2$ using metric duals. One can choose the basis in a unique way if the following conditions,

$$e^1 \wedge e^2 = \alpha + \beta \quad \text{for} \quad \alpha \in \Lambda^2_+,\ \beta \in \Lambda^2_-,\ |\alpha| = |\beta| = 1/\sqrt{2}$$

are imposed. Conversely, starting with a decomposable 2-form $\omega \in \Lambda^2$ i.e. $\omega = \theta \wedge \delta$ for some $\theta, \delta \in \Lambda^1$. The duals $\{\theta^\sharp, \delta^\sharp\}$ gives an oriented basis for a plane. In this correspondence, a complex plane correspond to a form in the form $\frac{\omega}{2} + \varphi$ for an anti-self-dual 2-form φ. If a plane $\tilde{\sigma}$ corresponds to (α, β) or simply $\sigma = \alpha + \beta \in \Lambda^2$,

$$\sec(\tilde{\sigma}) = \mathrm{Rm}(\sigma, \sigma) = \langle \mathcal{R}(\sigma), \sigma \rangle$$

where $\mathcal{R} : \Lambda^2 \to \Lambda^2$ is the curvature operator. Recall that for complex planes $\tilde{\sigma}, \tilde{\tau}$ we compute the *bisectional curvature* by

$$H(\tilde{\sigma}, \tilde{\tau}) = \mathrm{Rm}(\sigma, \tau).$$

Let us recall the decomposition of the curvature operator $\mathcal{R} : \Lambda^2 \to \Lambda^2$. If (M, g) is any oriented 4-manifold, then the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ implies that the curvature operator $\mathcal{R}$ can be decomposed as

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W^- + \frac{s}{12} \end{array} \right)$$

where $\Lambda^2_\pm$ stands for the self-dual and anti-self-dual 2-forms, i.e. $\Lambda^2_\pm = \{\phi \in \Lambda^2 : *\phi = \pm\phi\}$, where $*$ is the Hodge-$*$ operator determined by the metric g. $W^\pm$ is the self-dual/anti-self-dual Weyl curvature tensor, s is the scalar curvature and $\mathring{r}$ stands for the trace-free part of the Ricci curvature tensor r.

Now we are ready to state and prove our first estimate lemma. Let (M, J, g) be a compact Einstein-Hermitian 4-manifold. In this case, according to a theorem of LeBrun in [LeB97], such a space is either Kähler, or else they are conformally related to a Kähler metric $\tilde{g}$ with positive scalar curvature $\tilde{s}$ in such a way that $g = \tilde{s}^{-2}\tilde{g}$. Moreover, the non-Kähler Einstein-Hermitian metrics of Theorem 1.4, namely the Page metric and the Chen-LeBrun-Weber metric, are known to have (constant) positive scalar curvature. From now on, we will decorate the curvature expressions of the conformally related Kähler metric $\tilde{g}$ with tilde (e.g. $\tilde{s}, \tilde{R}, \tilde{W}$ etc.). We will denote the inner product of tensors with respect to $\tilde{g}$ by $\langle , \rangle_{\tilde{g}}$.

Lemma 4.1 (First Estimate)**.** *Let (M, J, g) be an Einstein-Hermitian metric which is not Kähler. Let $\tilde{g}$ be the conformally related Kähler metric such that $g = \tilde{s}^{-2}\tilde{g}$. If g has positive holomorphic bisectional curvature and λ is an eigenvalue of the operator $\tilde{W}_- : \Lambda^2_- \to \Lambda^2_-$, then we have*

$$\lambda < \frac{\tilde{s}}{6}.$$

Proof. Take two arbitrary unit tangent vector X, Y in T_pM at an arbitrary point p, and let $\varphi, \psi \in \Lambda^2_+$ be the anti-self-dual 2-forms of length $1/\sqrt{2}$ corresponding to the complex lines $\{X, JX\}$ and $\{Y, JY\}$, as described above. Keeping in mind that $\mathring{r} = 0$ since g is Einstein and Λ^2_+ and Λ^2_- are orthogonal with respect to g, we see by the decomposition of $\mathcal{R}$ that

$$\begin{aligned} \mathrm{H}(X,Y) &= \langle \mathcal{R}\left(\tfrac{\omega}{2} + \varphi\right) , \tfrac{\omega}{2} + \psi \rangle \\ &= \langle \left(W_+ + \tfrac{s}{12}\right) \tfrac{\omega}{2} + \left(W_- + \tfrac{s}{12}\right) \varphi, \tfrac{\omega}{2} + \psi \rangle \\ &= \tfrac{1}{4}\langle W_+\omega, \omega\rangle + \tfrac{s}{48}\langle \omega, \omega\rangle + \langle W_-\varphi, \psi\rangle + \tfrac{s}{12}\langle \varphi, \psi\rangle \\ &= \tfrac{1}{4}\langle W_+\omega, \omega\rangle + \tfrac{s}{12}\left\{\tfrac{1}{2} + \langle \varphi, \psi\rangle\right\} + \langle W_-\varphi, \psi\rangle. \end{aligned}$$

Now, since $g = \tilde{s}^{-2}\tilde{g}$, we see that

$$\langle W_+\omega, \omega\rangle = (W_+)_{ij}{}^{kl}\, \omega_{kl}\, \omega^{ij} = \tilde{s}^2(\tilde{W}_-)_{ij}{}^{kl}\; \tilde{s}^{-2}\tilde{\omega}_{kl}\; \tilde{s}^2\tilde{\omega}^{ij} = \tilde{s}^2\langle \tilde{W}_+\tilde{\omega}, \tilde{\omega}\rangle_{\tilde{g}}$$

It is well known [Der83] that for Kähler metrics the Kähler form is an eigenvector of the self-dual Weyl operator, more explicitly $\tilde{W}_+\tilde{\omega} = \frac{\tilde{s}}{6}\tilde{\omega}$. Thus we see that $\langle \tilde{W}_+\tilde{\omega}, \tilde{\omega}\rangle_{\tilde{g}} = \frac{\tilde{s}}{6}\langle \tilde{\omega}, \tilde{\omega}\rangle_{\tilde{g}} = \frac{\tilde{s}}{3}$ and hence

$$\mathrm{H}(X,Y) = \frac{\tilde{s}^3}{12} + \frac{s}{12}\left\{\frac{1}{2} + \langle \varphi, \psi\rangle\right\} + \langle W_-\varphi, \psi\rangle$$

Now, since the operator $\tilde{W}_- : \Lambda^2_- \to \Lambda^2_-$ is symmetric, it is diagonalizable. Let λ be an eigenvalue, $\varphi \in \tilde{E}_\lambda \subset \Lambda^2_-$ be a corresponding eigenvector with $|\varphi| = 1/\sqrt{2}$ (norm taken with respect to g). Choosing $\psi := -\varphi$ in the above equation yields,

$$\begin{aligned} \mathrm{H}(X,Y) &= \tfrac{\tilde{s}^3}{12} + \tfrac{s}{12}\left\{\tfrac{1}{2} - |\varphi|^2\right\} - \langle \tilde{s}^2\tilde{W}\varphi, \varphi\rangle \\ &= \tfrac{\tilde{s}^3}{12} + \tilde{s}^2\langle -\lambda\varphi, \varphi\rangle = \tfrac{\tilde{s}^3}{12} - \lambda\tilde{s}^2|\varphi|^2 \\ &= \tfrac{\tilde{s}^2}{2}\left\{\tfrac{\tilde{s}}{6} - \lambda\right\} > 0. \end{aligned}$$

and hence $\frac{\tilde{s}}{6} - \lambda > 0$, as required. $\square$

Lemma 4.2 (Second Estimate)**.** *Let (M, J, g) be an Einstein-Hermitian metric which is not Kähler. If g has positive holomorphic bisectional curvature, then for all $\varphi \in \Lambda^2_-$ with $|\varphi| = 1/\sqrt{2}$, we have*

$$\frac{\tilde{s}}{12} - \langle \tilde{W}_- \varphi, \varphi \rangle > 0.$$

Proof. Inserting $\psi = -\varphi$ into the first equation of the previous proof yields,

$$\begin{aligned} \mathrm{H}(\varphi, -\varphi) &= \tfrac{\tilde{s}^3}{12} + \tfrac{s}{12}\left\{\tfrac{1}{2} - |\varphi|^2\right\} - \langle W_- \varphi, \varphi \rangle = \tfrac{\tilde{s}^3}{12} - \langle \tilde{s}^2 W_- \varphi, \varphi \rangle \\ &= \tilde{s}^2 \left\{\tfrac{\tilde{s}}{12} - \langle \tilde{W}_- \varphi, \varphi \rangle\right\} > 0. \end{aligned}$$

□

The following Weitzenböck formula will be used in the proof of the main result which involves the Weyl curvature.

Theorem 4.3 (Weitzenböck Formula [Bou81])**.** *On a Riemannian manifold, the Hodge/modern Laplacian can be expressed in terms of the connection/rough Laplacian as*

$$(d + d^*)^2 = \nabla^* \nabla - 2W + \frac{s}{3}$$

where ∇ is the Riemannian connection and W is the Weyl curvature tensor.

Theorem 4.4. *Let M be a compact, Einstein-Hermitian 4-manifold of positive holomorphic bisectional curvature. Then the Betti number b^2_- vanishes.*

Here b^2_- stands for the number of negative eigenvalues of the cup product in $H^2(M)$ as usual. By Hodge theory, this number is equal to the dimension of anti-self-dual harmonic 2-forms on the Riemannian manifold (M, g).

Proof. These spaces are either Kähler, or else conformally Kähler by [LeB97]. In the first case, according to the resolution [SY80] of Frankel's conjecture by Siu and Yau, the space has to be complex projective plane. By Theorem 1.2, the metric has to be the Fubini-Study metric. In the latter case, we can write $g = \tilde{s}^{-2}\tilde{g}$ for g the Einstein, and $\tilde{g}$ the Kähler metric.

Assume, for a contradiction, there is a nonzero anti-self-dual 2-form $\varphi \in \Gamma(\Lambda^2_-)$ which is harmonic with respect to the Kähler metric $\tilde{g}$. Since we are on a compact manifold, we can rescale this form by a constant to have length strictly less than $1/\sqrt{2}$ everywhere. We write Weitzenböck Formula for the Kähler metric $\tilde{g}$:

$$0 = \Delta_{\tilde{g}} \varphi = \tilde{\nabla}^* \tilde{\nabla} \varphi - 2\tilde{W}\varphi + \frac{\tilde{s}}{3}\varphi$$

Now we take the L^2-inner product (with respect to $\tilde{g}$) of both sides with φ:

$$0 = \int_M \langle \tilde{\nabla}\varphi, \tilde{\nabla}\varphi\rangle_{\tilde{g}} - 2\langle \tilde{W}\varphi, \varphi\rangle_{\tilde{g}} + \frac{\tilde{s}}{3}\langle \varphi, \varphi\rangle_{\tilde{g}}\, d\mu_{\tilde{g}}$$
$$= \int_M |\tilde{\nabla}\varphi|^2_{\tilde{g}} + \tilde{s}^{-4}\left\{\frac{\tilde{s}}{6} - 2\langle \tilde{W}_-\varphi, \varphi\rangle_g\right\} d\mu_{\tilde{g}}.$$

At the points where $\varphi = 0$, the term in the curly bracket is just the scalar curvature term, hence strictly positive. Let p be a point where $\varphi|_p \neq 0$. Take an open set around p on which φ is nowhere zero. Now working on this open set, let $\tilde{\varphi} := \varphi/|\varphi|\sqrt{2}$ so that $|\tilde{\varphi}| = 1/\sqrt{2}$. Since the conformally related $\tilde{\varphi}$ is also anti-self-dual and of constant norm, the second estimate $\frac{\tilde{s}}{12} > \langle \tilde{W}_-\tilde{\varphi}, \tilde{\varphi}\rangle$ is applicable for $\tilde{\varphi}$:

$$\begin{aligned}\left\{\tfrac{\tilde{s}}{6} - 2\langle \tilde{W}_-\varphi, \varphi\rangle_g\right\}_p &= \tfrac{\tilde{s}}{6} - 2\langle \tilde{W}_-\sqrt{2}|\varphi|\tilde{\varphi}, \sqrt{2}|\varphi|\tilde{\varphi}\rangle_g \\ &= \tfrac{\tilde{s}}{6} - 4|\varphi|^2\langle \widetilde{W}_-\tilde{\varphi}, \tilde{\varphi}\rangle_g \\ &> \tfrac{\tilde{s}}{6} - 4.\,\tfrac{1}{2}\cdot\tfrac{\tilde{s}}{12}\ = 0\end{aligned}$$

at the point p. Since the term in parenthesis is strictly positive everywhere we get $\nabla\varphi = 0, \tilde{s}^{-4} = 0$, a contradiction. So that no such φ exists implying $b^2_- = 0$. $\square$

Remark 4.5. Theorem 4.4 is actually true under the weaker assumption that *orthogonal bisectional curvatures are positive*; that is, $H(X,Y) > 0$ whenever X is perpendicular to Y and JY. Indeed, in the non-Kähler case, when establishing the inequalities in the proof of Lemma 4.1 and Lemma 4.2, we only considered the case when $\varphi = -\psi$, which corresponds to taking two *orthogonal* complex lines span$\{X, JX\}$ and span$\{Y, JY\}$. On the other hand, in the Kähler-Einstein case, we notice that Theorem 1.2 remains true if we weaken the assumption to positivity of orthogonal bisectional curvatures. See the proof of Theorem 5 in [GK67], p.232, where the authors prove that the Einstein constant is a positive constant multiple of any of the orthogonal bisectional curvatures $H(X, JX)$, $H(X,Y)$ or $H(X, JY)$. We thank F. Belgun for this remark.

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References

[And11] David Andersson. *The 3-sphere and the bicycling black rings event horizon*. Thesis. Stockholm University, 2011.

[Ber65] Marcel Berger. Sur quleques variétés riemanniennes compactes d'Einstein. *C. R. Acad. Sci. Paris*, 260:1554–1557, 1965.

[Bou81] Jean-Pierre Bourguignon. Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d'Einstein. *Invent. Math.*, 63(2):263–286, 1981.

[Cal85] Eugenio Calabi. Extremal Kähler metrics. II. In *Differential geometry and complex analysis*, pages 95–114. Springer, Berlin, 1985.

[Der83] Andrzej Derdziński. Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.*, 49(3):405–433, 1983.

[DG91] Sorin Dragomir and Renata Grimaldi. Cauchy-Riemann submanifolds of locally conformal Kaehler manifolds. III. *Serdica*, 17(1):3–14, 1991.

[DO98] Sorin Dragomir and Liviu Ornea. *Locally conformal Kähler geometry*, volume 155 of *Progress in Mathematics*. Birkhäuser Inc., Boston, MA, 1998.

[EGH80] Tohru Eguchi, Peter B. Gilkey, and Andrew J. Hanson. Gravitation, gauge theories and differential geometry. *Phys. Rep.*, 66(6):213–393, 1980.

[FIP04] Maria Falcitelli, Stere Ianus, and Anna Maria Pastore. *Riemannian submersions and related topics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

[Fra61] Theodore Frankel. Manifolds with positive curvature. *Pacific J. Math.*, 11:165–174, 1961.

[GK67] Samuel I. Goldberg and Shoshichi Kobayashi. Holomorphic bisectional curvature. *J. Differential Geometry*, 1:225–233, 1967.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[KK15] Mustafa Kalafat and Caner Koca. Conformally Kähler surfaces and orthogonal holomorphic bisectional curvature. *Geom. Dedicata*, 174:401–408, 2015.

[KS16] Mustafa Kalafat and Ramazan Sarı. On special submanifolds of the Page space. *E-print available at arXiv:1608.03252. math.DG*, 2016.

[Koc12] Caner Koca. *On Conformal Geometry of Kähler Surfaces*. ProQuest LLC, Ann Arbor, MI. Ph.D. Thesis. Stony Brook University, 2012.

[Koc14] Caner Koca. Einstein Hermitian metrics of positive sectional curvature. *Proc. Amer. Math. Soc.*, 142(6):2119–2122, 2014.

[LeB97] Claude LeBrun. Einstein metrics on complex surfaces. In *Geometry and physics (Aarhus, 1995)*, volume 184 of *Lecture Notes in Pure and Appl. Math.*, pages 167–176. Dekker, New York, 1997.

[LeB12] Claude LeBrun. On Einstein, Hermitian 4-manifolds. *J. Differential Geom.*, 90(2):277–302, 2012.

[Pag78] Don N. Page. A Compact Rotating Gravitational Instanton. *Phys. Let.*, 79B(3):235–238, 1978.

[Pag09] Don N. Page. Some Gravitational Instantons. *E-print available at arXiv:0912.4922. physics.gr-qc*, 2009.

[SM99] Jean-François Sadoc and Rémy Mosseri. *Geometrical frustration*. Collection Aléa-Saclay: Monographs and Texts in Statistical Physics. Cambridge University Press, Cambridge, 1999.

[SY80] Yum Tong Siu and Shing Tung Yau. Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.*, 59(2):189–204, 1980.

[Via11] Jeff Viaclovsky. Lecture notes: Topics in riemannian geometry, 2011. URL: `http://www.math.wisc.edu/~jeffv/courses/865_Fall_2011.pdf`.

[Zha04] Xiao Zhang. Scalar flat metrics of Eguchi-Hanson type. *Commun. Theor. Phys. (Beijing)*, 42(2):235–238, 2004.