Algebraic topology of G_2 manifolds

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Abstract

In this paper we give a survey of various results about the topology of oriented Grassmannian bundles related to the exceptional Lie group G_2. Some of these results are new. We give self-contained proofs here. One often encounters these spaces when studying submanifolds of manifolds with calibrated geometries. For the sake of completeness we decided to collect them here in a self-contained way to be easily accessible for future usage in calibrated geometry. As an application we deduce existence of certain special 3 and 4 dimensional submanifolds of G_2 manifolds with special properties, which appear in the first named author's work with S. Salur about G_2 dualities.

1 Introduction

Recall that $G_2 \subset SO(7)$ is the 14-dimensional exceptional Lie group defined as the automorphisms of the imaginary octonions $im(\mathbb{O}) = \mathbb{R}^7$ preserving the cross product operation $\mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ (e.g. [HL], [Br], [AS1], [AS2]). Octonions are the elements of the 8 dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ where $\mathbb{H}$ are the quaternions, $\mathbb{O}$ is generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. The cross product operation $\times$ on $im(\mathbb{O})$ is induced from the octonion multiplication on $\mathbb{O}$ by $u \times v = im(\bar{v}.u)$. We say an oriented 7-manifold M^7 has a G_2 *structure* if its $SO(7)$- tangent frame bundle lifts to a G_2-bundle by the canonical fibration:

$$G_2 \to SO(7) \to \mathbb{RP}^7 \to BG_2 \to BSO(7).$$

Alternatively G_2 can be defined by the special 3-frames in $\mathbb{R}^7$ as follows:

$$G_2 = \{(u_1, u_2, u_3) \in (\mathbb{R}^7)^3 \mid \langle u_i, u_j \rangle = \delta_{ij},\ \langle u_1 \times u_2, u_3 \rangle = 0 \},$$

or as linear automorphisms of $\mathbb{R}^7$ preserving a certain 3-form $\varphi_0 \in \Omega^7(\mathbb{R}^7)$

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0 \}$$

where $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ with $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$.

By using this last definition, a G_2 structure on M^7 can be defined as a 3-form $\varphi \in \Omega^3(M^7)$ such that at each $p \in M$ the pair $(T_p(M), \varphi(p))$ is (pointwise) isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. This condition is equivalent to reducing the tangent frame bundle of a (not necessarily oriented) 7-manifold M from $GL(7, \mathbb{R})$ to G_2.

The form φ induces an orientation $\mu \in \Omega^7(M)$ on M, a metric $g = \langle , \rangle_\varphi$ by $\langle u, v\rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi]/6\mu$, and φ also defines a cross product operation $TM \times TM \mapsto TM$: $(u, v) \mapsto u \times v = u \times_\varphi v$ by $\varphi(u, v, w) = \langle u \times v, w\rangle$.

A manifold with G_2 structure (M^7, φ) is called a G_2 *manifold* (or an integrable G_2 structure) if at each point $p \in M$ there is an open chart $(U, p) \to (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of the open set U we can write $\varphi(x) = \varphi_0 + O(|x|^2)$. The condition that (M, φ) be a G_2 manifold is equivalent to φ being parallel under the induced metric connection $\nabla^\varphi(\varphi) = 0$, which turns out to be equivalent to the condition $d\varphi = d^*\varphi = 0$.

Let $G_k^+\mathbb{R}^n$ denote the Grassmannian manifold or oriented k-planes in $\mathbb{R}^n$. We call $L \in G_3^+\mathbb{R}^7$ an *associative* 3*-plane* if $\varphi|_L \equiv vol(L)$. A 3-dimensional submanifold $Y \subset (M, \varphi)$ is called *associative* if $\varphi|_Y \equiv vol(Y)$. An equivalent condition of a submanifold Y^3 to be associative is that $\chi|_Y \equiv 0$, where $\chi = \chi_\varphi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by $\langle \chi(u, v, w), z\rangle = *\varphi(u, v, w, z)$. This last identity implies a very useful property: χ assigns to every 3-plane $L \subset TM$ an orthogonal vector $\chi|_L \in L^\perp \subset TM$. We also have:

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2 = |u \wedge v \wedge w|^2.$$

$$\chi(u, v, w) = -u \times (v \times w) - \langle u, v\rangle w + \langle u, w\rangle v$$

We call $L \in G_3^+(\mathbb{R}^7)$ an *Harvey-Lawson* 3*-plane* (HL plane in short) if $\varphi|_L \equiv 0$. We call $S \in G_4^+(\mathbb{R}^7)$ a *coassociative* 4*-plane* if $\varphi|_S \equiv 0$. A 4-dimensional submanifold $X^4 \subset (M, \varphi)$ is called *coassociative* if $\varphi|_X = 0$. A manifold pair (X^4, Y^3) such that $Y^3 \subset X^4 \subset (M^7, \varphi)$ is called a *Harvey-Lawson pair* if the $\varphi \equiv 0$ on the restriction of the fibers of the normal bundle $\nu(X)|_Y$ of $X \subset (M, \varphi)$. The Grassmannians $G_3^+(\mathbb{R}^7)$ and $G_4^+(\mathbb{R}^7)$ have the following natural submanifolds

$$ASS_0 = \{L \in G_3^+(\mathbb{R}^7) \mid \varphi|_L = 0\}$$

$$ASS_+ = \{L \in G_3^+(\mathbb{R}^7) \mid \varphi|_L = vol(L)\}$$

$$ASS_- = \{L \in G_3^+(\mathbb{R}^7) \mid \varphi|_L = -vol(L)\}$$

$$COASS = \{S \in G_4^+(\mathbb{R}^7) \mid \varphi|_S = 0\}$$

When there is no danger of confusion, we will abbreviate ASS_+ by ASS. Note that there is a natural identification $ASS \approx COASS$ given by $L \mapsto L^\perp$, and also

$$ASS_\pm \approx G_2/SO(4)$$

$$ASS_0 \approx G_2/SO(3)$$

From these descriptions it follows that ASS_0 is a sphere bundle over $ASS_\pm$

$$S^3 \to ASS_0 \to ASS_\pm$$

These special Grassmann manifolds sit in $G_3^+(\mathbb{R}^7)$ as level sets of the function

$$\Phi : G_3^+(\mathbb{R}^7) \to \mathbb{R}$$

given by $L = u \wedge v \wedge w \mapsto \varphi_0(u, v, w)$, where $\{u, v, w\}$ is an orthonormal basis of L.

We have $\Phi^{-1}(0) = ASS_0$ and $\Phi^{-1}(\pm 1) = ASS_\pm$, since φ_0 is calibrating 3-form $|\varphi_0(L)| \leq 1$. In this way $G_3^+(\mathbb{R}^7)$ appears as the double of the D^4-disk bundle over ASS. Φ is a Bott-Morse function. So that $ASS_\pm$ becomes two critical submanifolds with indices 0 and 4 respectively ([Z]).

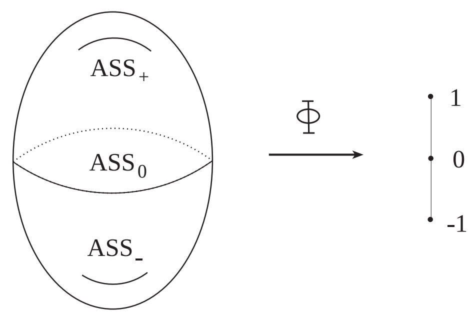

Figure 1: The map $\Phi : G_3^+(\mathbb{R}^7) \to \mathbb{R}$

These Grassmannians occur as the fibers of some bundles over 7-manifolds with G_2 structure (M^7, φ), providing a useful tool studying deformations of associative submanifolds [AS2]. Next we summarize some of the constructions from [AS1]. Let $\mathbb{P}_{SO(7)} \to M$ be frame bundle of the tangent bundle $T(M) \to M$ of any closed smooth oriented 7-manifold M, and let $\widetilde{M} \to M$ be the bundle oriented 3-planes in TM, which is defined by the identification $[p, L] = [pg, g^{-1}L] \in \widetilde{M}$:

$$G_3^+(\mathbb{R}^7) \to \widetilde{M} = \mathbb{P}_{SO(7)}(M) \times_{SO(7)} G_3^+(\mathbb{R}^7) \xrightarrow{\pi} M.$$

Let $\xi \to G_3^+(\mathbb{R}^7)$, and $\nu = \xi^\perp \to G_4^+(\mathbb{R}^7)$ be the universal $\mathbb{R}^3$ bundle, and its dual $\mathbb{R}^4$ bundle, respectively. Therefore, $Hom(\xi, \nu) = \xi^* \otimes \nu \longrightarrow G_3^+(\mathbb{R}^7)$ is the tangent bundle $TG_3^+(\mathbb{R}^7)$. ξ, ν extend fiberwise to give bundles $\Xi \to \widetilde{M}$, $\mathbb{V} \to \widetilde{M}$ respectively. If Ξ^* be the dual of Ξ, then $Hom(\Xi, \mathbb{V}) = \Xi^* \otimes \mathbb{V} \to \widetilde{M}$ is the bundle of vertical vectors $T^v(\widetilde{M})$ of $T(\widetilde{M}) \to M$, i.e. the tangents to the fibers of π.

When (M^7, φ) is a manifold with G_2 structure, similarly to the construction above, we can form the following subbundles of $\widetilde{M} \to M$

$$ASS_\pm \to \mathbb{A}SS_\pm = \mathbb{P}_{G_2}(M) \times_{G_2} ASS_\pm \longrightarrow M$$

$$ASS_0 \to \mathbb{A}SS_0 = \mathbb{P}_{G_2}(M) \times_{G_2} ASS_0 \longrightarrow M$$

where $\mathbb{P}_{G_2}(M)$ is the G_2 frame bundle of the tangent bundle of (M, φ). In particular $\mathbb{A}SS = \mathbb{P}_{G_2}(M)/SO(4) \longrightarrow \mathbb{P}_{G_2}(M)/G_2 = M$. As in the previous section, the restriction of the universal bundles $\xi,\ \nu = \xi^{\perp} \to G_3^+(\mathbb{R}^7)$ induce 3 and 4 plane bundles $\Xi \to \mathbb{A}SS$ and $\mathbb{V} \to \mathbb{A}SS$. Also we have the similar map $L \to \varphi(L)$

$$\Phi : \widetilde{M} \longrightarrow \mathbb{R}$$

with $\Phi^{-1}(0) = \mathbb{A}SS_0$ and $\Phi^{-1}(\pm 1) = \mathbb{A}SS_{\pm}$. Fiberwise this is just the map previously described on $G_3^+(\mathbb{R}^7)$, it is the bundle version of the map described in Figure 1. So we have disjointly embedded pair of codimension 4-submanifolds $\mathbb{A}SS_{\pm} \subset \widetilde{M}$, which are separated by a codimension zero submanifold $\mathbb{A}SS_0 \subset \widetilde{M}$.

Any embedding of a 3-manifold $f : Y^3 \hookrightarrow M^7$, by its tangential Gauss map, lifts to an embedding $f_T : Y \hookrightarrow \widetilde{M}$ such that the pull-backs $f_T^*\Xi = T(Y)$ and $f_T^*\mathbb{V} = \nu(Y)$ are the tangent and normal bundles of Y. In particular, if f is an embedding of an associative submanifold , then the image of f_T lands in $\mathbb{A}SS$

$$\begin{array}{ccc} & & \widetilde{M} \supset \mathbb{A}SS_0,\ \mathbb{A}SS_{\pm} \\ & f_T \nearrow & \downarrow \\ Y & \xrightarrow{f} & M \end{array}$$

Similarly any embedding of a 4-manifold $f : X^4 \hookrightarrow M^7$, by its normal Gauss map, induces an embedding $f_N : X \hookrightarrow \widetilde{M}$, such that $f_N^*\Xi = \nu(X)$ and $f_N^*\mathbb{V} = T(X)$ are the normal and tangent bundles of X^4.

If $\mathbb{L} = \Lambda^3(\Xi) \to \widetilde{M}$ is the determinant (real) line bundle. By the discussion above χ maps every oriented 3-plane in $T_x(M)$ to its 4-dimensional complementary subspace, so χ gives a bundle map $\mathbb{L} \to \mathbb{V}$ over $\widetilde{M}$, which is a section of $\mathbb{L}^* \otimes \mathbb{V} \to \widetilde{M}$. Since Ξ is oriented $\mathbb{L}$ is trivial, so χ actually gives a section

$$\chi = \chi_{\varphi} \in \Omega^0(\widetilde{M}, \mathbb{V})$$

$\mathbb{A}SS$ is the zeros of this section. Associative submanifolds $Y \subset M$ are characterized by the condition $\chi|_{\tilde{Y}} = 0$, where $\tilde{Y} \subset \widetilde{M}$ is the canonical lifting of Y.

$\mathbb{A}SS$ is the universal space parameterizing associative submanifolds of M. In particular, if $\tilde{f} : Y \hookrightarrow \widetilde{M}_{\varphi}$ is the lifting of an associative submanifold, by pulling back we see that the principal $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \to \widetilde{M}_{\varphi}$ induces an $SO(4)$-bundle $\mathcal{P}(Y) \to Y$, and gives the following vector bundles via the representations:

$$\begin{array}{lcl} \nu(Y) & : & y \mapsto qy\lambda^{-1} \\ T(Y) & : & x \mapsto qxq^{-1} \end{array} \tag{1}$$

where $[q, \lambda] \in SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$, $\nu = \nu(Y)$ and $T(Y) = \lambda_+(\nu)$. Here we use quaternionic notation $\mathbb{R}^4 = \mathbb{Q}$ and $\mathbb{R}^3 = im(\mathbb{Q})$. Also we can identify T^*Y with TY by the induced canonical metric coming from G_2 structure. From above we have the action $T^*Y \otimes \nu \to \nu$ inducing actions $\Lambda^*(T^*Y) \otimes \nu \to \nu$.

In the next section we survey relevant results from algebraic topology of these spaces. Most of these results are elementary, folklore or already known (e.g. [Bo],

[Mi], [SZ]), but some results and the application in Section §3 are new. There are also other applications e.g. [KÜ]. For the sake of completeness we decided to collect them here in a self contained way to be easily accessible for future usage in calibrated geometry.

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2 Algebraic topology of Grassmann bundles

Here compute cohomology groups of various Grassmann bundles. To this end we first start with the following calculations.

Lemma 2.2. *The homology of the Grassmann manifold $G_2^+\mathbb{R}^7$ of oriented 2-planes is given by the following table:*

$$H_*(G_2^+\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z},0,\mathbb{Z},0,\mathbb{Z},0,\mathbb{Z},0,\mathbb{Z},0,\mathbb{Z}).$$

This result is obtained by some elementary computations on various spectral sequences. After this, by computing the torsion and using the Gysin sequence of some special fibrations we get the following.

Theorem 2.7. *The homology of the oriented Grassmann manifold $G_3^+\mathbb{R}^7$ is given by:*

$$H_*(G_3^+\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z},0,\mathbb{Z}_2,0,\mathbb{Z}\oplus\mathbb{Z},\mathbb{Z}_2,\mathbb{Z}_2,0,\mathbb{Z}\oplus\mathbb{Z},\mathbb{Z}_2,0,0,\mathbb{Z}).$$

Also the cup product structure of the cohomology can be computed as follows.

Theorem 2.10. *The cohomology ring of $G_3^+\mathbb{R}^7$ is given by:*

$$H^*(G_3^+\mathbb{R}^7;\mathbb{Z}) = \mathbb{Z}[x,y]/(x^4, x^3-y^3, x^2-y^2, xy-yx)\oplus\mathbb{Z}_2[z,t]/(z^3,t^2,zt+tz)$$

where the degrees of the generators are $|x| = |y| = 4, |z| = 3$ and $|t| = 7$.

As a by product, along the way we compute the homology of a Stiefel manifold.

Theorem 2.5. *The homology of the Stiefel manifold $V_3\mathbb{R}^7$ is given by:*

$$H_*(V_3\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z},0,0,0,\mathbb{Z},\mathbb{Z}_2,0,0,0,\mathbb{Z}_2,0,\mathbb{Z},0,0,0,\mathbb{Z}).$$

Then by combining two different fibrations we can compute the homology of the Lie group G_2, and compute its cohomology ring as well.

Theorem 2.17. *The homology groups of the Lie group G_2 are given as follows:*

$$H_*(G_2;\mathbb{Z}) = (\mathbb{Z},0,0,\mathbb{Z},0,\mathbb{Z}_2,0,0,\mathbb{Z}_2,0,0,\mathbb{Z},0,0,\mathbb{Z}).$$

Theorem 2.18. *The cohomology ring of the Lie group G_2 can be described as follows.*

$$H^*(G_2;\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_{11}] \oplus \Lambda_{\mathbb{Z}_2}[x_6, x_9]/(x_6 x_9)$$

where the degrees of the generators are $|x_k| = k$.

In part 2.1 we deal with topological computations on the Grassmannian, in part 2.2 we compute the cup product structure. Finally, in part 2.3 we analyze the space G_2 and compute its cohomology ring, and then in 2.4 compute cohomology rings of of certain bundles associated to G_2 manifolds.

2.1 Homology of Grassmannians

The aim of this section is to compute some integral homology groups of the oriented real Grassmannian $G_3^+\mathbb{R}^7$. We will use various forms of the Serre spectral sequence of a fiber bundle with various coefficients. As a warm up, let us recall the homology and cohomology of the basic spaces. Starting from SO_3, using its identification with $\mathbb{RP}^3$ and a combination of Poincaré duality with the universal coefficients theorem (UCT) one easily computes its homology groups as

$$H_*(SO_3;\mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}).$$
$$H_*(SO_3;\mathbb{Z}_2) = (\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2).$$

Now, consider the Stiefel manifold $V_3\mathbb{R}^7$ which is defined to be the space of orthonormal 3-frames in $\mathbb{R}^7$. By the Stiefel fibrations (e.g. [Ha]) this is a 7–3–1=3 connected space. Since the dimension is even the fourth homotopy group is $\mathbb{Z}$ by Stiefel [Sti] so $\pi_{01234}(V_3\mathbb{R}^7) = (0, 0, 0, 0, \mathbb{Z})$. This notation expresses the homotopy groups of the space up to level 4. See [Wh, Pa] for higher homotopy. Sending a 3-frame to the oriented 3-plane, which it spans, gives us the fibration

$$SO_3 \longrightarrow V_3\mathbb{R}^7 \longrightarrow G_3^+\mathbb{R}^7. \tag{2}$$

Using the related homotopy exact sequence (HES) and homotopy groups

$$\pi_{01234}(SO_3) = (0, \mathbb{Z}_2, 0, \mathbb{Z}, \mathbb{Z}_2)$$
$$\pi_{01234}(G_3^+\mathbb{R}^7) = (0, 0, \mathbb{Z}_2, 0, \mathbb{Z} \oplus \mathbb{Z}).$$

Our next aim is to compute some homology groups for the Grassmannian. From above by the Hurewicz isomorphisms

$$H_{012}(G_3^+\mathbb{R}^7) = (\mathbb{Z}, 0, \mathbb{Z}_2).$$

The Poincaré polynomial of $G_3^+\mathbb{R}^7$ is known to be

$$p_{G_3^+\mathbb{R}^7}(t) = 1 + 2t^4 + 2t^8 + t^{12}. \tag{3}$$

See [GHV] vol.III, pp.494-496 for computations also [GMM]. For further homology computations, we will consult to the spectral sequences and the Gysin sequence. We will abbreviate $G = G_3^+\mathbb{R}^7$ and $V = V_3\mathbb{R}^7$ frequently in what follows. Our first assertion is the following Lemma.

Lemma 2.1. *For the oriented Grassmann manifold we have $H_3(G_3^+\mathbb{R}^7;\mathbb{Z}) = 0$.*

Proof. We consider the homological Serre spectral sequence with $\mathbb{Z}$-coefficients associated to the fiber bundle (2), properties of which is given as follows [Sa].

$$E^2_{p,q} := H_p(G; H_q(SO_3;\mathbb{Z}))$$

$$E^\infty_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where $F_{p,q}$ are abelian groups forming a filtration satisfying

$$0 = F_{-1,n+1} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = H_n(V;\mathbb{Z}).$$

The differentials are bidegree $(-n, n-1)$ maps

$$d^n : E^n_{p,q} \longrightarrow E^n_{p-n,q+(n-1)}.$$

Some of the terms appear in the following table. Note that

Table 1: *Homological Serre spectral sequence for $G_3^+\mathbb{R}^7$, second page.*

E^2						
3						
2	0					
1		0				
0				$H_3(G;\mathbb{Z})$		
	0	1	2	3	4	5

$$E^2_{1,1} = H_1(G; H_1(SO_3;\mathbb{Z})) = H_1(G;\mathbb{Z}_2) = H_1 \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(H_0;\mathbb{Z}_2) = 0.$$

We have immediate convergence for the $E^2_{3,0}$ term so that

$$H_3(G;\mathbb{Z}) = E^2_{3,0} = \cdots = E^\infty_{3,0} = F_{3,0}/F_{2,1} = H_3(V;\mathbb{Z})/F_{2,1} = 0.$$

□

Another result on the Grassmannian is the following.

Lemma 2.2. *The homology of the oriented Grassmann manifold $G_2^+\mathbb{R}^7$ is given by:*

$$H_*(G_2^+\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}).$$

Proof. Consider the fibration

$$S^1 = SO_2 \longrightarrow V_2\mathbb{R}^7 \longrightarrow G_2^+\mathbb{R}^7. \tag{4}$$

The homology of the Stiefel manifold is well-known (e.g. [Ha]), it is given by:

$$H_*(V_2\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z}, 0, 0, 0, 0, \mathbb{Z}_2, 0, 0, 0, 0, 0, \mathbb{Z}) \tag{5}$$

Since $V_2\mathbb{R}^7$ is $7-2-1=4$ connected, homotopy exact sequence of the above fibration immediately gives $\pi_{012}(G_2^+\mathbb{R}^7) = (0,0,\mathbb{Z})$. The homological Serre spectral sequence reads as follows.

$$E^2_{p,q} := H_p(G_2^+\mathbb{R}^7; H_q(S^1;\mathbb{Z}))$$

$$E^\infty_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where $F_{p,q}$ are abelian groups of the filtration

$$0 = F_{-1,n+1} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = H_n(V_2\mathbb{R}^7;\mathbb{Z}).$$

We first fill out the limiting page of the sequence as in the Table 2, except the terms in quotation marks, which we do not use, however the reader can compute them after secondary steps and written here for recording purposes only. Vanishing of the homology of $V_2\mathbb{R}^7$ and the filtrations easily handle this far.

Table 2: *The limiting page of Homological Serre spectral sequence for $G_2^+\mathbb{R}^7$.*

E^∞	0	1	2	3	4	5	6	7	8	9	10
1	0	0	0	0	'$\mathbb{Z}_2$'	0	0	0	0	0	
0		0	0	0	0	'0'	0	0	0	0	0

Next we fill out the second page. Keep in mind that in the following the entries of a column are identical. Columns till the second one follows from the homotopy groups. The third column is zero since we have the immediate convergence $E^2_{3,0} = E^\infty_{3,0}$ because of the differential. The fourth column is the outcome of the isomorphism

$$\mathbb{Z} \xleftarrow{\sim} E^2_{4,0} \ : \ d^2_{4,0}$$

Since the domain and image group have to converge to zero in the next page, this map is both injective and surjective. The sixth column and on are the consequences of the universal coefficients theorem and the Poincaré duality. Fifth column is the remaining one. Again, because of the immediate convergence $E^2_{5,1} = E^\infty_{5,1}$ the $E^2_{5,1}$ term has to vanish, so is this column. A row of Table 3

Table 3: *The second page.*

E^2	0	1	2	3	4	5	6	7	8	9	10
1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
0		0	$\mathbb{Z}$	0	$\mathbb{Z}$		$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

determines the homology of $G_2^+\mathbb{R}^7$ by its definition. □

Now we can prove our main Lemma.

Lemma 2.3. *The torsion subgroup of $H_4(G_3^+\mathbb{R}^7;\mathbb{Z})$ is trivial.*

Proof. Since the free part F_5 of $H_5(G;\mathbb{Z})$ is trivial 3, we can compute the torsion part of $H_4(G;\mathbb{Z})$ which is denoted by T_4 using cohomology as follows.

$$H^5(G;\mathbb{Z}) = \mathrm{Hom}(H_5,\mathbb{Z}) \oplus \mathrm{Ext}(H_4,\mathbb{Z}) = 0 \oplus T_4 = T_4.$$

To figure out this group we will work with two new fibrations.

$$\begin{tikzcd}
 & S^4 \arrow[d] & \\
S^2 \arrow[r] & S(E_3\mathbb{R}^7) = E_0 \arrow[r, "\pi_h"] \arrow[d, "\pi_v"] & G_3^+\mathbb{R}^7 \\
 & G_2^+\mathbb{R}^7 &
\end{tikzcd}$$

Here $E_3\mathbb{R}^7$ denotes the tautological bundle over $G_3^+\mathbb{R}^7$. The horizontal fibration is clear, which is obtained by removing the zero section. From there one can obtain the vertical fibration with the following procedure. A point in $G_2^+\mathbb{R}^7$ represents a 2- plane which is contained in $(7-2)(3-2) = 4$ parameter of 3-planes. Since we take the orientations into consideration we obtain spheres rather than projective spaces. One may think in terms of the oriented flag variety $F_{2,3}^+(\mathbb{R}^7)$ with its projection maps. See [Har]. Now, consider the Gysin exact sequence [MS] of the vertical fibration.

$$\cdots \longrightarrow H^0(G_2^+\mathbb{R}^7;\mathbb{Z}) \xrightarrow{\cup e_5} H^5(G_2^+\mathbb{R}^7;\mathbb{Z}) \xrightarrow{\pi_v^*} H^5(E_0;\mathbb{Z}) \longrightarrow H^1(G_2^+\mathbb{R}^7;\mathbb{Z}) \longrightarrow \cdots$$

Since we proved that the odd homology of the grassmannian $G_2^+\mathbb{R}^7$ is zero in Lemma 2.2, this implies that the middle term $H^5(E_0;\mathbb{Z})$ vanishes.
Next consider the Gysin sequence of the horizontal fibration.

$$\cdots \longrightarrow H^2(G_3^+\mathbb{R}^7;\mathbb{Z}) \xrightarrow{\cup e_3} H^5(G_3^+\mathbb{R}^7;\mathbb{Z}) \xrightarrow{\pi_h^*} H^5(E_0;\mathbb{Z}) \longrightarrow \cdots$$

Since $H_2(G_3^+\mathbb{R}^7;\mathbb{Z})$ is torsion and $H_1(G_3^+\mathbb{R}^7;\mathbb{Z})$ is zero we have $H^2(G_3^+\mathbb{R}^7;\mathbb{Z}) = 0$. Together with the vanishing of $H^5(E_0;\mathbb{Z})$ we obtain our result

$$H^5(G_3^+\mathbb{R}^7;\mathbb{Z}) = 0.$$

□

Since by 3 the fourth Betti number of $G_3^+\mathbb{R}^7$ is 2, this Lemma implies the following.

Corollary 2.4. *For the Grassmann manifold we have $H_4(G_3^+\mathbb{R}^7;\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}$.*

The results so far helps us to consume most of the homology of our Grassmann manifold. The homology at the levels 7 and above are easily deduced from the Ext universal coefficients theorem and Poincaré duality. Finally using (3) in addition yields that the homology in the levels 5 and 6 are solely torsion, isomorphic and denoted by T_5. The rest of this section is devoted to compute this group.

In order to compute this torsion this time we need some results on some Stiefel manifolds.

Theorem 2.5. *The homology of the Stiefel manifold $V_3\mathbb{R}^7$ is computed as,*

$$H_*(V_3\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z},0,0,0,\mathbb{Z},\mathbb{Z}_2,0,0,0,\mathbb{Z}_2,0,\mathbb{Z},0,0,0,\mathbb{Z}).$$

Proof. We will be using the homological Serre spectral sequence related to the following new fibration.

$$S^4 \longrightarrow V_3\mathbb{R}^7 \longrightarrow V_2\mathbb{R}^7.$$

This is obtained by projecting onto the first two vectors of the frame and the third one has unit independency in $\mathbb{R}^5$. Defining groups are as follows.

$$E^2_{p,q} := H_p(V_2\mathbb{R}^7; H_q S^4)$$

$$E^\infty_{p,q} = F_{p,q}/F_{p-1,q+1}$$

where $F_{p,q}$ are abelian groups of the filtration

$$0 = F_{-1,n+1} \subset \cdots \subset F_{n-1,1} \subset F_{n,0} = H_n(V_3\mathbb{R}^7;\mathbb{Z}).$$

Merely knowing the homology of $V_2\mathbb{R}^7$ as in (5) one can construct the second page of the spectral sequence. Because of the abundance of zeros and freeness

Table 4: *Homological Serre spectral sequence for $V_3\mathbb{R}^7$ with $\mathbb{Z}$-coefficients.*

$E^2 = E^\infty$

4	$\mathbb{Z}$					$\mathbb{Z}_2$						$\mathbb{Z}$
3												
2												
1												
0	$\mathbb{Z}$					$\mathbb{Z}_2$						$\mathbb{Z}$
	0	1	2	3	4	5	6	7	8	9	10	11

of $\mathbb{Z}$ we have the immediate convergence. Isomorphism of the groups in the filtration gives the triviality of the homology of $V_3\mathbb{R}^7$ at the levels 1 to 3, 6 to 8, 10 and 12 to 14 since their diagonal consist entirely of zeros. Now, to determine the homology at the 4th level start at $F_{-1,5} = 0$. Since $E^\infty_{0,4} = F_{0,4}/F_{-1,5} = \mathbb{Z}$ implying that $F_{0,4} \approx \mathbb{Z}$. Next

$$F_{1,3}/F_{0,4} = E^\infty_{1,3} = \cdots = E^\infty_{4,0} = F_{4,0}/F_{3,1} = 0$$

implying the isomorphisms

$$\mathbb{Z} \approx F_{0,4} \approx F_{1,3} \approx F_{2,2} \approx F_{3,1} \approx F_{4,0} = H_4(V_3\mathbb{R}^7;\mathbb{Z}).$$

In an exactly similar way the other two nontrivial groups on the 4th row projects onto the homology at levels 9 and 15 isomorphically, hence these are also determined. In the 5th level starting at $F_{-1,4} = 0$, the vanishing of the diagonal from top till $E^{\infty}_{4,1} = F_{4,1}/F_{3,2}$ forces the vanishing of the filtration till and including $F_{4,1}$. Now the limiting information $\mathbb{Z}_2 = E^{\infty}_{5,0} = F_{5,0}/F_{4,1}$ determines the 5th homology. Similarly the 11th level can be handled. □

Lemma 2.6. *For the Grassmann manifold we have the following.*

$$H_5(G_3^+\mathbb{R}^7;\mathbb{Z}) = H_6(G_3^+\mathbb{R}^7;\mathbb{Z}) = \mathbb{Z}_2.$$

Proof. We saw that these two groups are solely torsion and isomorphic to one another and denoted both of them by T_5. We will be working on the cohomological Serre spectral sequence with integer coefficients related to the fiber bundle (2), definition and limit of which is given as follows [Sa].

$$E_2^{p,q} := H^p(G\,;H^q(SO_3;\mathbb{Z}))$$

$$E_\infty^{p,q} = F^{p,q}/F^{p+1,q-1}$$

where $F^{p,q}$ are abelian groups forming a filtration satisfying

$$H^n(V;\mathbb{Z}) = F^{0,n} \supset F^{1,n-1} \supset \cdots \supset F^{n+1,-1} = 0.$$

The differentials are of bidegree $(n, -n+1)$ so satisfying

$$d_n : E_n^{p,q} \longrightarrow E_n^{p+n,q-(n-1)}.$$

Table 5: *Cohomological Serre spectral sequence for $G_3^+\mathbb{R}^7$, second page.*

	3	0	T_5	T_5				
	2			$T_{5,2}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$		
	1				0	0	0	
E_2	0						$\mathbb{Z}_2$	0
		5	6	7	8	9	10	11

Our first claim is that $F^{7,3} \approx \mathbb{Z}_2$ for this sequence. Using the Theorem 2.5 and the filtration we obtain the isomorphisms

$$\mathbb{Z}_2 = H_5 \approx H^{10}(V_3\mathbb{R}^7;\mathbb{Z}) = F^{0,10} \supseteqq F^{1,9} \supseteqq \cdots \supseteqq F^{7,3} \tag{6}$$

Table 6: *Limiting page. Underlined terms are hypothetical.*

	3		$\mathrm{Ker}d_2$				
	2		$\mathrm{Ker}d_3$	$\underline{\mathbb{Z}_2^2}$	0		
	1				0		
E_∞	0					$\underline{\mathbb{Z}_2}$	
		6	7	8	9	10	11

provided by the vanishing of the limiting entries $E_\infty^{0,10}, \cdots, E_\infty^{6,4}$. Since we have

$$\mathrm{Ker}d_2 = E_\infty^{7,3} = F^{7,3}/F^{8,2}$$

the only two possibilities 0 or $\mathbb{Z}_2$ are remaining for $\mathrm{Ker}d_2$. After this point let us assume that $T_5 = 0$ to raise a contradiction. Table 6 shows the limit under this hypothesis. Note that the underlined terms are purely hypothetical. The two facts

$$\mathbb{Z}_2 \approx F^{7,3} \supset F^{8,2} \quad \text{and} \quad \mathbb{Z}_2^2 = E_\infty^{8,2} = F^{8,2}/F^{9,1}$$

shows that the group $\mathbb{Z}_2^2$ is way large to be carried by the filtration (6). So that we now know $T_5 \neq 0$. Next we claim that the torsion group T_5 is solely 2-torsion. To see it use the fundamental theorem of finitely generated abelian groups [DF] to conclude that this group is a direct sum of $\mathbb{Z}_{p^k}$'s for prime numbers $p \geq 2$ not necessarily distinct. If one of the p's is odd than by the partial converse to Lagrange theorem (or the Sylow's theorem) there is a subgroup of order p. This subgroup is contained in the $\mathrm{Ker}d_2$ otherwise its image would be a group of order 2 which has to divide p. However none of the two possibilities of $\mathrm{Ker}d_2$ above covers a subgroup of odd order. So T_5 is a direct sum of $\mathbb{Z}_{2^k}$'s. These summands are cyclic, so pick a generator i.e. an element so that the order $|a| = p^k$. Next we claim that k cannot be greater than or equal to 3. If that is the case to minimize the kernel d_2 must be surjective, in any case $|\mathrm{Ker}d_2| \geq 2^k/2 = 2^{k-1} \geq 2$ causes a problem. So that $k \leq 2$ hence T_5 can consist of $\mathbb{Z}_2$ or $\mathbb{Z}_4$ summands only. Computing the following entry of the spectral sequence

$$E_2^{7,2} = H^7(G;\mathbb{Z}_2) = \mathrm{Hom}(H_7,\mathbb{Z}_2) \oplus \mathrm{Ext}(H_6,\mathbb{Z}_2) = \mathrm{Ext}(T_5,\mathbb{Z}_2)$$

and denoting this term by $T_{5,2}$ (it counts the number of even ordered irreducible summands). We note that $T_{5,2}$ cannot hope to survive till infinity since $H^9(V;\mathbb{Z}) = F^{0,9}$ is trivial. So that the map

$$d_3 : E_3^{7,2} = T_{5,2} \longrightarrow E_3^{10,0} = \mathbb{Z}_2$$

is injective. Reminding ourselves that $\mathrm{Ext}(\mathbb{Z}_4,\mathbb{Z}_2) = \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) = \mathbb{Z}_2$, the outcome is $T_{5,2} \subset \mathbb{Z}_2$. Since T_5 is nonzero we have $T_{5,2} = \mathbb{Z}_2$. Hence T_5 is either $\mathbb{Z}_2$ or $\mathbb{Z}_4$. To raise a contradiction, suppose $T_5 = \mathbb{Z}_4$. Then $\mathrm{Ker}d_2$ cannot be zero by the cardinality, the remaining possibility is Ker$d_2 \approx \mathbb{Z}_2$ by above. Now we pass to the diagonal on the left. The differential

$$d_2^{6,3} : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

cannot be surjective if it were, that would raise an isomorphism of the cyclic and Klein 4-group. So that it has a cokernel denoted $Cok = E_\infty^{8,2}$ of order 2 or 4. Now concentrating on the $10th$ diagonal, our assumption

$$\mathrm{Ker} d_2 \approx \mathbb{Z}_2 = E_\infty^{7,3} = F^{7,3}/F^{8,2}$$

together with the fact that $F^{7,3} \approx \mathbb{Z}_2$ would imply that $F^{8,2} = 0$. Consequently we would obtain $E_\infty^{8,2} = 0$, a contradiction. □

We can now collect the results of this section to obtain the following.

Theorem 2.7. *The homology of the oriented Grassmann manifold $G_3^+\mathbb{R}^7$ is computed as,*

$$H_*(G_3^+\mathbb{R}^7;\mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z} \oplus \mathbb{Z}, \mathbb{Z}_2, 0, 0, \mathbb{Z}).$$

2.2 Cup product structure

In this section we will analyze the cup product structure of the Grassmann manifold $G_3^+\mathbb{R}^7$. We start with the free part. We will be using and interpreting the computations in [SZ]. Recall ξ and ν denote the *canonical (3-plane) bundle* and its orthogonal complement 4-plane bundle on this space respectively. Denoting the first Pontryagin and Euler classes of these bundles by $p = p_1(\xi)$ and $e = e(\nu)$ actually, we have the following.

Theorem 2.8. *The exterior algebra of the Grassmannian manifold $G_3^+\mathbb{R}^7$ is given as follows*

$$H^*_{dR}(G_3^+\mathbb{R}^7) = \mathbb{R}[p,e]/(e^4, p^3 - e^3, p^2 - e^2, pe - ep) \ \text{ for } \ |p| = |e| = 4.$$

Proof. Reading the Theorem 7.5 of [SZ], $p \pm e$ are generators of the fourth cohomology with Poincaré duals $2[ASS_+], 2[ASS_-]$. And reading Theorem 7.4, p^2, pe generate the eighth cohomology. The de Rham integral $\langle e^3, [G_3^+\mathbb{R}^7]\rangle = 2$ settles a non-zero class so that e^3 generates the top cohomology. So that the additive structure is given as follows.

$$H^*_{dR}(G_3^+\mathbb{R}^7) = \langle p+e, p-e\rangle \oplus \langle p^2, pe\rangle \oplus \langle e^3\rangle.$$

Among the relations $e^2 = p^2$ is given in Section 7,

$$\begin{aligned}
\langle pe^2, [G_3^+\mathbb{R}^7]\rangle &= \langle pe(e+p)/2 + pe(e-p)/2, [G_3^+\mathbb{R}^7]\rangle \\
&= \tfrac{1}{2}\langle pe\,, 2[ASS]\rangle + \tfrac{1}{2}\langle pe\,, -2[\widetilde{ASS}]\rangle \\
&\overset{7.4}{=} 2.
\end{aligned}$$

□

Theorem 2.9. *The torsion algebra of the Grassmannian manifold $G_3^+\mathbb{R}^7$ is given as follows*

$$\mathbb{Z}_2[x_3, x_7]/(x_3^3, x_7^2, x_3x_7 + x_7x_3) \text{ for } \quad |x_3| = 3, |x_7| = 7.$$

Proof. There is torsion at four levels $3, 6, 7, 10$ as we have computed in section 2.1. To understand the cup product structure we consult to the cohomological Serre spectral sequence with $\mathbb{Z}$-coefficients. Table 7 shows the second third page of

Table 7: *Cohomological Serre spectral sequence for $G_3^+\mathbb{R}^7$. Second page.*

	3	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$
	2	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	1	0	0	0	0	0	0	0	0	0	0	0	0	0
E_2	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}^2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$
		0	1	2	3	4	5	6	7	8	9	10	11	12

this spectral sequence. In Table 8 for the third page of the spectral sequence, the arrows are isomorphism as follows. We have $H^3(V;\mathbb{Z}) = 0 = F^{0,3}$ from Theorem 2.5 implying that the limit term $E_\infty^{3,0} = F^{3,0}/F^{4,-1}$ vanishes. To provide that the only incoming non-zero differential $d_3 : E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism. The second one is similar, the vanishing of $E_\infty^{3,2} = F^{3,2}/F^{4,1}$ is guaranteed through the vanishing of $H^5(V;\mathbb{Z}) = F^{0,5}$. The last one is achieved through $H^9(V;\mathbb{Z}) = 0 = F^{0,9}$ hence $E_\infty^{7,2} = 0$.

Table 8: *Cohomological Serre spectral sequence for $G_3^+\mathbb{R}^7$.*

	2	$\mathbb{Z}_2 a$			$\mathbb{Z}_2 a x_3$	$\mathbb{Z}_2^2$		0	$\mathbb{Z}_2 a x_7$			
	1											
E_3	0	$\mathbb{Z}1$			$\mathbb{Z}_2 x_3$			$\mathbb{Z}_2 x_6$	$\mathbb{Z}_2 x_7$		0	$\mathbb{Z}_2 x_{10}$
		0	1	2	3	4	5	6	7	8	9	10

Following the techniques in [Ha2] we label the generators as above. Replacing some generators with their negatives if necessary we may assume $d_3 a = x_3$. Similarly we may assume $d_3(ax_3) = x_6$. Combining with the following relation

$$d_3(ax_3) = d_3 a x_3 + a d x_3 = x_3^2 + a0 = x_3^2$$

we replace $x_6 = x_3^2$. Again assuming $d_3(ax_7) = x_{10}$ and applying the following identity

$$d_3(ax_7) = d_3 a x_7 + a d x_7 = x_3 x_7$$

we replace $x_{10} = x_3x_7$. Since there is no cohomology at the level 9, $x_3^3 = 0$. One can alternatively see this as follows. Since $H^8(V;\mathbb{Z}) = 0 = F^{0,8}$ we have $E_\infty^{6,2} = 0$

forces that the entry $E_3^{6,2} = 0$. From the isomorphism $E_3^{0,2} \otimes E_3^{6,0} \longrightarrow E_3^{6,2}$ we get $ax_6 = ax_3^2 = 0$. Taking the differential of both sides yields the result.

$$0 = d_3(ax_3^2) = d_3(a)x_3^2 + ad_3(x_3^2) = x_3^3.$$

Last relation follows from the alternating property of the cup product for odd dimensions. $\square$

Combining the two results we obtain the following.

Theorem 2.10. *The cohomology ring of the Grassmannian manifold $G_3^+\mathbb{R}^7$ is given as follows*

$$H^*(G_3^+\mathbb{R}^7;\mathbb{Z}) = \mathbb{Z}[x,y]/(x^4, x^3 - y^3, x^2 - y^2, xy - yx) \oplus \mathbb{Z}_2[z,t]/(z^3, t^2, zt + tz)$$

where the degrees are $|x| = |y| = 4, |z| = 3$ and $|t| = 7$, and $x = e(\nu)$, $y = p_1(\xi)$.

Next we would like to see how the submanifold of associative planes sits inside $G_3^+\mathbb{R}^7$ cohomologically. We would like to mention that the space of associative 3-planes and its Stiefel-Whitney classes are studied to some degree by [BH]. We have the pullback map induced by the inclusion

$$H^q(ASS;\mathbb{Z}) \longleftarrow H^q(G_3^+\mathbb{R}^7;\mathbb{Z}) : i^*$$

which operates at the levels $q = 0 \cdot\cdot 8$. The nonzero integral cohomology groups are already computed in Section 10 of [SZ] to be the following.

$$H^q(ASS;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & q = 0,4,8 \\ \mathbb{Z}_2, & q = 3,6 \end{cases}$$

We can compute the cohomological ring and the action of the inclusion map on cohomology as follows.

Theorem 2.11. *We have the following facts for the 8-manifold of associative planes.*

(a) The cohomology ring structure is given as

$$H^*(ASS;\mathbb{Z}) = \mathbb{Z}[d]/\langle d^3\rangle \oplus \mathbb{Z}_2[c]/\langle c^3\rangle$$

where the degrees are $|d| = 4$ and $|c| = 3$.

(b) The inclusion map $i : ASS \to G_3^+\mathbb{R}^7$ acts on the cohomology rings as follows

$$i^*x = i^*y = d, \;\; i^*z = c, \;\; i^*t = 0.$$

Proof. The proof that we will give for the two parts are somehow interrelated.

Table 9: *Cohomological Serre spectral sequence for $G_2^+\mathbb{R}^7$.*

		0	1	2	3	4	5	6	7	8
	2	$\mathbb{Z}a$			$\mathbb{Z}_2ax_3$	$\mathbb{Z}ax_4$		$\mathbb{Z}_2ax_6$		$\mathbb{Z}ax_8$
	1									
$E_3 = E_2$	0	$\mathbb{Z}1$			$\mathbb{Z}_2x_3$	$\mathbb{Z}x_4$		$\mathbb{Z}_2x_6$		$\mathbb{Z}x_8$

1. We will employ the fibration $S^3 \to G_2^+\mathbb{R}^7 \to ASS$, see [SZ] for the map. Corresponding Serre cohomological spectral sequence yields the following page. Here c appears as the pullback of the Euler class of the tautological bundle on the Grassmannian which is non-zero. Labeling the generators in the spectral sequence as in Table 9, the surjective map yields the relation $d_3a = x_3$. Then the isomorphism implies $x_6 = d_3(ax_3) = d_3ax_3 \pm adx_3 = x_3^2$. This shows that c^2 is also a nonzero element, hence the generator of its level. Another relation is obtained through the third map $0 = d_3(ax_4) = x_3x_4$. Relabel $d = x_4$.

2. Next we will deduce that the element d^2 is the generator of its level. And this will finish the proof on the part (a). To see this let

$$H^4(ASS;\mathbb{Z}) = \langle d\rangle \text{ and } H^8(ASS;\mathbb{Z}) = \langle s\rangle.$$

   Then $d^2 = \alpha\, s$ for some $\alpha \in \mathbb{Z}$. Also suppose that $i^*x = \beta\, d$ and $i^*y = \gamma\, d$ for some $\beta, \gamma \in \mathbb{Z}$. Recall that additively we have the following generators at the level 8 of the Grassmannian,

$$H^8(G_3^+\mathbb{R}^7;\mathbb{Z}) = \langle p_1^2E, p_1EeF\rangle$$

   where $x = p_1E$ and $y = eF$. By the Lemma 7.4 of [SZ] we have the following integrals

$$\langle x^2, [ASS]\rangle = \langle xy, [ASS]\rangle = 1,$$

   so that these elements map onto the generator d^2 of the eighth cohomology of ASS. Elaborating this fact by

$$\begin{aligned} s &= i^*x^2 = i^*(xy) \\ &= \beta^2\, d^2 = \gamma^2\, d^2 \end{aligned}$$

   and plugging in yields

$$s = \beta^2\alpha\, s.$$

   So that $\beta^2\alpha = 1$. This forces $\alpha = 1$ hence the assertion.

3. Since now we have obtained $s = d^2$, we have $|\beta| = |\gamma| = 1$. We can assume that $\beta = 1$ after a change of sign of the generator d if needed. We claim

the same for γ as well, suppose $\gamma = -1$ to raise a contradiction. Then $i^*(x+y) = 0$ and this implies

$$\begin{aligned} 0 &= i^*(x+y)^2 \\ &= i^*(x^2+xy+yx+y^2) \\ &= d^2+2i^*(xy)+d^2 \end{aligned}$$

yields the contradiction $i^*(xy) = -d^2$ to facts of the previous part. Combining these we have $i^*x = i^*y = d$.

4. The element c comes naturally as the restriction of the Euler class z of the tautological vector bundle E^7_3. Finally there is no seventh cohomology to map onto.

□

2.3 The Lie Group G_2

In this section we will compute some invariants of the Lie Group G_2, which is defined to be the subgroup of SO_7 which fixes the 4-form,

$$*\phi_0 = dx^{4567} + dx^{2367} + dx^{2345} + dx^{1357} - dx^{1346} - dx^{1256} - dx^{1247}.$$

Here the notation dx^{4567} suggest $dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7$ similarly the others. In order to work efficiently on G_2 we will need to use two fibrations first of which is the following.

$$SU_3 \longrightarrow G_2 \longrightarrow S^6 \tag{7}$$

Since G_2 consists of orthogonal transformations it preserves the sphere in $\mathbb{R}^7$ so that it has an action on the 6-sphere. This is a transitive action and the stabilizer of a point on the sphere, preserves its orthogonal complement as well hence a subgroup of SO_6. See a general reference [Br] for further details.

To work with the fibration we need the cohomology of the fiber. One can obtain the cohomology of the unitary group as an exterior algebra

$$H^*(U_n;\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_1, x_3 \cdots x_{2n-1}]$$

using the complex Stiefel manifolds. Then via the action of the special unitary group on the unitary one, the fibration $SU_n \to U_n \to S^1$ helps to drop the first generator and we get

$$H^*(SU_3;\mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_5] = (\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \mathbb{Z}, 0, 0, \mathbb{Z}).$$

Now we can work on the cohomological Serre spectral sequence for the first fibration (7) which up to the sixth page looks like in Table 10. Note that the real dimension of the Lie algebra $\mathfrak{su}_3$ is computed to be 8 so that G_2 is 14 dimensional. We can immediately compute from this sequence that there is no cohomology at

Table 10: *Cohomological Serre spectral sequence for the SU_3 fibration of G_2.*

	8	$\mathbb{Z}$						$\mathbb{Z}$
	7							
	6							
	5	$\mathbb{Z}$						$\mathbb{Z}$
	4							
	3	$\mathbb{Z}$						$\mathbb{Z}$
	2							
	1							
$E_2 = E_6$	0	$\mathbb{Z}$						$\mathbb{Z}$
		0	1	2	3	4	5	6

the levels $1, 2, 4, 7, 10, 12, 13$ and there is a $\mathbb{Z}$ each at the levels $3, 11$. This is more or less the only accessible information to get at first sight from this fibration, this is mainly because we do not use the actual fibration map which could have been the trivial product $SU_3 \times S^6$ as well. Since this is a fibration the Euler characteristic is multiplicative and

$$\chi(G_2) = \chi(SU_3)\chi(S^6) = 0 \cdot 2$$

so that this implies $b_8 = b_5$. At this point using universal coefficients and Poincaré duality,

$$H^5 = H^8 = F_5 \text{ and } H^6 = H^9 = F_5 \oplus T_5.$$

is the ultimate statement for the missing cohomology along with the following Lemma. Here F_5 and T_5 denotes the free and torsion part of the fifth homology.

Lemma 2.12. *The fifth Betti number $b_5(G_2) \leq 1$.*

Proof. From the filtration one can show that

$$H^9(G_2; \mathbb{Z}) \approx F^{6,3} \approx E_\infty^{6,3} = \mathbb{Z}/\operatorname{Im} d_6^{0,8}.$$

A similar computation can be done to see $H^6(G_2; \mathbb{Z}) \approx \mathbb{Z}/\operatorname{Im} d_6^{0,5}$. Since we know that these two are isomorphic, as a by product one can easily see the equality $\operatorname{Im} d_6^{0,8} = \operatorname{Im} d_6^{0,5}$. □

To recover the missing information we consult to the second fibration (9). To analyze this we need the homology of the fiber. We use the fibration

$$SO_3 \longrightarrow SO_4 \longrightarrow S^3 \tag{8}$$

and applying the homological spectral sequence as in the Table 11 yields the homology,

$$H_*(SO_4; \mathbb{Z}) = (\mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}^2, \mathbb{Z}_2, 0, \mathbb{Z}).$$

Table 11: *Homological Serre spectral sequence for the SO_3 fibration of SO_4.*

	3	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
	2	0	0	0	0
	1	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$
$E^2 = E^\infty$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
		0	1	2	3

Here one should use universal coefficients to compute the homology at level 4 and to show it is free at level 3. Then computing the Euler characteristic $\chi(SO_4) = \chi(SO_3)\chi(S^3)$ to be zero yields $b_3 = 2$, hence the result.
The second fibration of G_2 is as follows.

$$SO_4 \longrightarrow G_2 \longrightarrow ASS \tag{9}$$

To see this, note that the group $G_2 \subset SO_7$ naturally acts on $\mathbb{R}^7$, leaves the associative form invariant. So that an associative (three) plane is sent to another associative plane under G_2 action, hence the set ASS of associative planes stays invariant. The stabilizer of this action is the orthogonal transformations of $\mathbb{R}^7$ which leaves an associative 3-plane invariant, and hence acts orthogonally in the complement yielding SO_4. According to the rule

Table 12: *Homological Serre spectral sequence for the SO_4 fibration of G_2.*

	6	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$
	5	0	0	0	0	0	0	0	0	0
	4	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
	3	$\mathbb{Z}^2$	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}_2^2$	0	0	$\mathbb{Z}^2$
	2	0	0	0	0	0	0	0	0	0
	1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
E^2	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}$
		0	1	2	3	4	5	6	7	8

$$E^2_{p,q} = H_p(ASS; H_q(SO_4; \mathbb{Z}))$$

Table 12 shows the starting page for the homological Serre spectral sequence for this fibration. The following is our first main assertion.

Lemma 2.13. *The free part F_5 is zero hence $b_5(G_2) = 0$.*

Proof. We actually claim that the line $p + q = 9$ at infinity is totally zero. To see this consider the filtration

$$0 = F_{-1,10} = F_{0,9} = \cdots = F_{4,5} \subset \cdots \subset F_{7,2} \subset F_{8,1} = F_{9,0} = H_9(G_2;\mathbb{Z}) = F_5.$$

Over the line at infinity the only possibly nonzero terms are $E^\infty_{5,4}$ and $E^\infty_{8,1}$ which both are subgroups of $\mathbb{Z}_2$. Since lower terms at infinity are zero we have $E^\infty_{5,4} = F_{5,4}$. So that $F_{5,4}$ is a subgroup of a free group and a subgroup of $\mathbb{Z}_2$ at the same time. So it has to vanish. Then if you follow up zeros till $(8,1)$ you can do the same argument to see that $F_{8,1} = 0$, hence the result. $\square$

This gives us a chance to say something about the torsion.

Lemma 2.14. *The torsion part $T_5 = \mathbb{Z}_m$ for some $m \geq 2$ for the Lie group G_2. In particular it is nonzero.*

Proof. Recall from Lemma 2.12 that we have

$$H^9(G_2;\mathbb{Z}) \approx \mathbb{Z}/\operatorname{Im} d_6^{0,8} = F_5 + T_5 = T_5.$$

Assume that $d_6^{0,8}$ is surjective or $T_5 = 0$ to raise a contradiction. In that case passing to the second (homological) sequence where $H_8(G_2;\mathbb{Z}) = 0$ and the $p + q = 8$ line disappears in the limit. In particular $E^\infty_{5,3} = 0$. The only differential from or hitting $(5,3)$ are $d^2_{5,3}$ and $d^3_{8,1}$. $d^2_{5,3}$ should better be surjective and $d^3_{8,1}$ better be an embedding to bleed $E^2_{5,3} = \mathbb{Z}_2^2$ to nothing since these two are the only two chances. The second assertion means

$$E^4_{8,1} := \operatorname{Ker} d^3_{8,1} = 0.$$

But now consider the unpleasant situation for $E^2_{4,4} = \mathbb{Z}_2$ which has to bleed into death. Only possibly nontrivial differential is the following

$$d^4_{8,1} : E^4_{8,1} \longrightarrow E^4_{4,4}$$

which emanates from zero as we computed, a contradiction. $\square$

We will also be using the following Lemma.

Lemma 2.15. *We have that $E^\infty_{2,6} = E^\infty_{8,0} = 0$ for the limits. Moreover $E^3_{5,4} = E^3_{2,6} = 0$. Hence only possibly nonzero terms on the line $p + q = 8$ at infinity are $E^\infty_{4,4}$ and $E^\infty_{5,3}$.*

Proof. Since we know that $E^\infty_{5,4} = 0$, the differential

$$d^2_{5,4} : E^2_{5,4} \longrightarrow E^2_{2,6}$$

has to be injective hence an isomorphism, enough to kill the entry $(2,6)$. Moreover this implies also that $E^3_{5,4} = E^3_{2,6} = 0$ on both parts. Since $E^2_{8,0} = \mathbb{Z}$ is free, so is $E^\infty_{8,0} = F_{8,0}/F_{7,1}$. Considering $F_{8,0} = H_8(G_2;\mathbb{Z}) = T_5 = \mathbb{Z}_m$, $E^\infty_{8,0}$ also torsion hence trivial. $\square$

Finally we can now handle the torsion piece.

Lemma 2.16. *The torsion part $T_5 = \mathbb{Z}_2$ for the Lie group G_2.*

Proof. All of the terms from $E^\infty_{0,8}$ till and including $E^\infty_{3,5}$ vanish. On the filtration this implies that

$$0 = F_{-1,9} = F_{0,8} = F_{1,7} = F_{2,6} = F_{3,5} \subset F_{4,4} \subset \cdots \subset F_{8,0} = H_8(G_2;\mathbb{Z}) = T_5 = \mathbb{Z}_m$$

the left hand terms vanish and $E^\infty_{4,4} \approx F_{4,4}$. From the starting entry of the spectral sequence we have $E^\infty_{4,4} \trianglelefteq \mathbb{Z}_2$ hence $F_{4,4} \trianglelefteq \mathbb{Z}_2$ as well. So we have only two possibilities for $F_{4,4}$. We will analyze these two cases separately.

Case 1: Assume $F_{4,4} = \mathbb{Z}_2$. Then all differentials related to the $(4,4)$ terms are zero for its survival. In particular the differential

$$d^4_{8,1} : E^4_{8,1} \longrightarrow E^4_{4,4}$$

is zero. That implies the convergence $E^4_{8,1} = E^\infty_{8,1} = 0$. Since from Lemma 2.15 we have $0 = E^3_{2,6} = E^6_{2,6}$ so that the differential $d^6_{8,1} : E^6_{8,1} \longrightarrow E^6_{2,6}$ is zero. Consequently the only nonzero differential emanating from the entry $(8,1)$ is the

$$d^3_{8,1} : E^3_{8,1} = \mathbb{Z}_2 \longrightarrow E^3_{5,3} = K \tag{10}$$

which has to be injective to provide $E^\infty_{8,1} = 0$. Here, by definition we take $K := \operatorname{Ker} d^2_{5,3}$ so that this kernel has a subgroup of order two, in particular it is nontrivial. In the following we will show that on the other end $K \neq \mathbb{Z}_2^2$ as well. To see this observe that the differential

$$d^5_{8,0} : E^5_{8,0} \lhd \mathbb{Z} \longrightarrow E^5_{3,4} \lhd \mathbb{Z}_2$$

has to be zero. If it does not then it would mean that $E^5_{8,0} \approx \mathbb{Z}$. But this term vanishes ultimately by Lemma 2.15 and there is no chance to vanish since the differential can no longer be injective. The outgoing differential $d^3_{3,4} : \mathbb{Z}_2 \longrightarrow \mathbb{Z}$ is zero. So the only possibly nontrivial differential concerning $(3,4)$ is

$$d^2_{5,3} : E^5_{5,3} = \mathbb{Z}_2^2 \longrightarrow E^2_{3,4} = \mathbb{Z}_2$$

has to be surjective to kill it since $H_7(G_2;\mathbb{Z}) = 0$ so that it disappears at infinity. To be surjective the kernel K cannot be everything. So we reach at the only possibility that $K \approx \mathbb{Z}_2$. But the differential at (10) was injective now becomes an isomorphism. That kills the term and we get $E^\infty_{5,3} = 0$. That tells the result as follows

$$\mathbb{Z}_2 = F_{4,4} = F_{5,3} = F_{6,2} = F_{7,1} = F_{8,0} = \mathbb{Z}_m.$$

Case 2: Assume $F_{4,4} = 0$. Then $E^{\infty}_{4,4} = F_{4,4}/F_{3,5} = 0$. Then the only possibly nontrivial differential incoming or emanating from $E^{*}_{4,4}$ is

$$d^4_{8,1} : E^4_{8,1} \longrightarrow E^4_{4,4} = \mathbb{Z}_2$$

which has to be surjective. So $E^4_{8,1} = \mathbb{Z}_2$, implying that $E^3_{8,1} = \mathbb{Z}_2$ as well and the map

$$d^4_{8,1} : E^3_{8,1} = \mathbb{Z}_2 \longrightarrow E^3_{5,3}$$

is zero. Letting $K := \operatorname{Ker} d^2_{5,3}$ then we have $E^{\infty}_{5,3} = K$. The elements on the filtration become

$$F_{5,3} = F_{6,2} = F_{7,1} = F_{8,0} = \mathbb{Z}_m,$$

so that

$$K = E^{\infty}_{5,3} = F_{5,3}/F_{4,4} = F_{5,3} = \mathbb{Z}_m.$$

We also know that $K \trianglelefteq \mathbb{Z}_2^2$ so that $\mathbb{Z}_m \trianglelefteq \mathbb{Z}_2^2$ and it $m \geq 2$ implies that $m = 2$.

□

Accumulating the results of this section we get the following.

Theorem 2.17. *The homology groups of the Lie group G_2 are as follows.*

$$H_*(G_2; \mathbb{Z}) = (\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z}, 0, 0, \mathbb{Z}).$$

At this point we are in a position to understand the cup product structure of G_2 as follows.

Theorem 2.18. *The cohomology ring of the Lie group G_2 can be described as follows.*

$$H^*(G_2; \mathbb{Z}) = \Lambda_{\mathbb{Z}}[x_3, x_{11}] \oplus \Lambda_{\mathbb{Z}_2}[x_6, x_9]/(x_6 x_9)$$

where the degrees are $|x_k| = k$.

Proof. Consider Table 13 that presents all possible generators. The maps are multiplication by 2 which are injective and so that x_5, x_8 disappears. Since the image is zero, the product

$$E_6^{0,3} \otimes E_6^{0,3} \longrightarrow E_6^{0,6}$$

is zero which is the same as cup product upto a sign. So that this implies $x_3^2 = 0$. Likewise one can compute $x_6^2 = x_9^2 = x_{11}^2 = 0$. Since

$$E_6^{0,3} = H^0(S^6; H^3(SU_3; \mathbb{Z})) \approx H^3(SU_3; \mathbb{Z})$$

the product

$$E_6^{0,3} \otimes E_6^{6,5} \longrightarrow E_6^{6,8}$$

is just multiplication of coefficients. Hence the multiplication $x_3 \cdot : E_6^{6,5} \longrightarrow E_6^{6,8}$ is an isomorphism. In particular sends generators to generators hence $x_3 \cdot x_{11} = x_{14}$ if the signs are arranged suitably. The only missing relation

$$x_3 x_{11} = (-1)^{3 \cdot 11} x_{11} x_3$$

comes by the properties of the cup product. □

Table 13: *Cohomological Serre spectral sequence for the SU_3 fibration of G_2.*

	8	$\mathbb{Z}x_8$		$\mathbb{Z}x_{14}$
	7		$\times 2$	
	6			
	5	$\mathbb{Z}x_5$		$\mathbb{Z}x_{11}$
	4		$\times 2$	
	3	$\mathbb{Z}x_3$		$\mathbb{Z}x_9$
	2			
	1			
$E_2 = E_6$	0	$\mathbb{Z}1$		$\mathbb{Z}x_6$
		0	1 2 3 4 5	6

2.4 Classifying Space

In this section we will compute the classifying space BG_2 of the group G_2. We will be using the results [Bo] of A. Borel. See [Mi] for a recent exposition. We start with the free algebra. A theorem of Borel tells us the following. Let G be a compact, connected Lie group and R be $\mathbb{Z}$ or a field k of characteristic p. Assume $H_*(G)$ is torsion free if $R = \mathbb{Z}$, or is p-torsion free if $R = k$. Then there are universally transgressive elements $x_i \in H^{n_i}(G; R)$ such that

$$H^*(G;R) = \Lambda[x_1 \cdots x_l] \text{ where } |x_i| = n_i \text{ odd},$$

$$H^*(BG;R) = R[y_1 \cdots y_l] \text{ where } y_i = \tau(x_i).$$

Here $\tau : H^k(BG) \to H^{k+1}(G)$ is the transgression map of the fiber bundle $G \to EG \to BG$. This is a co-analogue of the connecting homomorphism of the homotopy exact sequence of a fiber bundle. A corollary of this theorem is that if $H_*(G)$ is p-torsion free, then $H_*(BG)$ is also p-torsion free. In our case this means that BG_2 has no p-torsion for $p \geq 3$. At this point, taking a $\mathbb{Z}_3$ coefficient cohomology ring kills the (2) torsion and captures the free piece by universal coefficients theorem. Starting with the free part of G_2 as we computed in Theorem 2.18 we have generators at the levels 3 and 11. Transgression increases the degree by one and applying the above we have

$$H^*(BG_2; \mathbb{Z}_3) = \mathbb{Z}_3[y_4, y_{12}].$$

For the torsion part, by the application of the above theorems, it is well-known that the cohomology with $\mathbb{Z}_2$ coefficients is

$$H^*(G_2;\mathbb{Z}_2) = \mathbb{Z}_2[x_3]/(x_3^4) \otimes \Lambda[x_5] \text{ where } x_5 = Sq^2 x_3,$$

$$H^*(BG_2;\mathbb{Z}_2) = \mathbb{Z}_2[y_4, y_6, y_{10}] \text{ where } y_6 = Sq^2 y_4 \text{ and } y_7 = Sq^3 y_4 = Sq^1 y_6.$$

Now, comparing the generators, in this $\mathbb{Z}_2$ coefficient ring, y_4 comes from the free part, y_6, y_{10} are new so that they are produced by the torsion.

Theorem 2.19. *The cohomology ring of the classifying space of the Lie group G_2 is*

$$H^*(BG_2;\mathbb{Z}) = \mathbb{Z}[y_4, y_{12}] \oplus \mathbb{Z}_2[y_6, y_{10}]$$

where the degrees are $|y_k| = k$.

3 Existence of Harvey-Lawson pairs

Here we illustrate an application of the topological results which we have proved in the previous section. By applying the Leray-Hirsch theorem (e.g. [Sp], [Ha]) Theorem 2.10 can be generalized from $G_3^+\mathbb{R}^7$ to the Grassmann bundle $\pi : \widetilde{M} \to M$. This is because of the fact that the Euler and Pontryagin classes $x = e(\nu)$, $y = p_1(\xi)$ are restrictions of the Euler and Pontryagin classes $e(\mathbb{V})$, $p_1(\Xi)$ of the corresponding universal bundles over $\widetilde{M}$ (cohomological extension property).

Theorem 3.1. *$H^*(\widetilde{M};\mathbb{Q})$ is an $H^*(M;\mathbb{Q})$ module generated by $e(\mathbb{V})$ and $p_1(\Xi)$. In other words the map*

$$a \otimes x + b \otimes y \mapsto \pi^*(a) \cup e(\mathbb{V}) + \pi^*(b) \cup p_1(\Xi)$$

gives an isomorphism:

$$H^*(M) \otimes_{\mathbb{Z}} H^*G_3^+(\mathbb{R}^7) \longrightarrow H^*(\widetilde{M}).$$

Next comes a corollary to the existence of Harvey-Lawson pairs. Recall that a manifold pair $Y^3 \subset X^4 \subset (M^7, \varphi)$ in a manifold with G_2-structure is called a *Harvey-Lawson pair* if the three form vanishes on the normal bundle of X^4 when restricted to Y^3. Note that these type of submanifolds are related to the Mirror-duality of [AS3] and [AS4].

Corollary 3.2. *Let $X^4 \hookrightarrow (M^7, \varphi)$ be any embedding of a closed smooth 4-manifold into a manifold with G_2 structure satisfying the property $\langle p_1(\nu X), [X]\rangle \neq \pm e[X]$, where $e[X]$ is the Euler characteristic. After a small isotopy of $X \subset M$, we can find a nonempty closed smooth 3-dimensional submanifold $Y^3 \subset X^4$ such that (X, Y) is a HL pair.*

Proof. Consider the map $\Psi : Im(X, M) \times X \longrightarrow \widetilde{M}$ given by $\Psi(f, x) = f_N(x)$, where $Im(X, M)$ denotes the space of embeddings of X into M. f_N assigns the normal plane to $f(X) \subset M$ at the image of a point $x \in X$. By transversality (e.g. [GP]) we can find a nearby isotopic copy f of any embedding, such that $f_N : X \to \widetilde{M}$ is transverse to the submanifolds $\mathbb{A}SS_- \sqcup \mathbb{A}SS_+ \sqcup \mathbb{A}SS_0$. Since $p_1(\nu X) \pm e(TX) \neq 0$, f_N meets both $\mathbb{A}SS_\pm$ since their Poincaré duals are $p_1(\Xi) \pm e(\mathbb{V})$. Hence $Y^3 := f^{-1}(\mathbb{A}SS_0) \neq 0$, and by definition (X, Y) is a HL-pair. □

Example 3.3. *By this corollary one can produce examples of HL pairs. Standard embeddings of S^4, $S^2 \times S^2$ (after stabilization) into $\mathbb{R}^7$ can be extended to an embedding into any manifold with G_2 structure via a coordinate diffeomorphism. Since these have trivial*

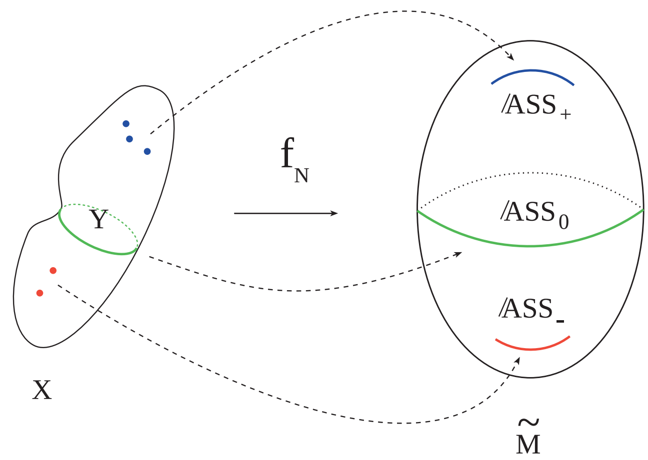

Figure 2: The map $\Phi : G_3^+\mathbb{R}^7 \to \mathbb{R}$

normal bundle and nontrivial Euler characteristic, they satisfy the hypothesis of the corollary and can be isotoped to a HL pair. Similarly an orientable closed surface Σ_g of genus g is embedded into $\mathbb{R}^3$ with trivial normal bundle. So that $\Sigma_g \times \Sigma_h$ can be embedded into any manifold with G_2 structure for $g \neq 1, h \neq 1$ and isotoped to a HL pair.

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