

Minimal submanifolds and stability in Einstein manifolds

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To the memory of Jim Simons

Abstract

In this paper, we compute the index and nullity for minimal submanifolds of some complex Einstein spaces. We investigate the stability of these minimal submanifolds and suggest a criterion for instability for some cases. We also compute some higher eigenvalues for the Laplacian of the Berger spheres and provide with an algorithm.

Keywords: Einstein metrics; Hermitian metrics; minimal submanifold; Laplacian.

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1 Introduction

Minimal submanifolds are an intriguing and fundamental concept in differential geometry, characterized by the property that they locally minimize area. These submanifolds are critical points for the area functional, meaning their mean curvature vector vanishes at every point. The study of minimal submanifolds has profound implications in various fields, including geometric analysis, mathematical physics, and general relativity. In the 20th century, the field experienced significant advances. Simons' work bridged differential geometry and global analysis, significantly influencing the study of minimal submanifolds.

This paper investigates minimal submanifolds within specific Einstein spaces, notably the complex projective plane CP_2 endowed with the Fubini-Study metric, the Page space as its blow-up with the Einstein-Hermitian metric. Our focus is on calculating the indices and nullities of these submanifolds, providing a deeper understanding of their geometric properties. The concept of index, introduced by [Sim68], is defined as the sum of the dimensions of the eigenspaces corresponding to the negative eigenvalues of the Jacobi operator. Simons also introduced the notion of nullity, defined as the dimension of the eigenspace corresponding to the zero eigenvalue of the Jacobi operator. In his seminal work, Simons computed these quantities for minimal submanifolds in round spheres, establishing the following key result.

Theorem 1.1. [Sim68] *A totally geodesic sphere S^p immersed in a round S^n has*

$$\text{index} = n - p \quad \text{and} \quad \text{nullity} = (n - p)(p + 1).$$

He also proceeds to prove the following lower bounds on the index and nullity of a general minimal submanifold.

Theorem 1.2. [Sim68] *Let M be a p -dimensional, closed, minimal submanifold immersed in S^n . Then the following bounds hold.*

1. *The index of M is greater than or equal to $(n - p)$, and the equality holds only when M is S^n .*
2. *The nullity of M is greater than or equal to $(n - p)(p + 1)$, and the equality holds only when M is S^n .*

There exist further advancements on the minimal submanifolds of spheres in the late 20th century. Among them, building up on the foundations of Simons, Ejiri provided better estimates for the index of minimal 2-spheres in higher-dimensional spheres.

Theorem 1.3. [Eji83] *Let M be a closed orientable minimal surface of genus zero immersed in S^{2n} for $n > 2$. Then the index of M is greater than or equal to $2(n(n + 2) - 3)$.*

In this paper, we studied the index and nullity of the Berger sphere on three different Kähler manifolds, which have never been investigated in this aspect before. For this purpose, we introduce a theorem on the Jacobi operator of totally geodesic submanifolds in Einstein manifolds.

Theorem 2.1. *Let $f : M \rightarrow \bar{M}$ be a totally geodesic immersion of a p -manifold into an Einstein n -manifold. The following are true.*

1. *The Jacobi operator of M is equal to $J = -\Delta_M - (s/n)I + Ric^\perp$.*
2. *If M is a hypersurface or \bar{M} is of constant curvature, then $J = -\Delta_M - (s/n)I$.*

Then we study the spectrum of the Laplacian on the related Riemannian spaces, and developed an idea to calculate the eigenvalues of the Jacobi operator. Along the way we also compute some higher eigenvalues for the Laplacian of the Berger spheres and provide with an algorithm. We have made the following observation.

Theorem 4.2. *The Berger sphere in $\mathbb{C}P_2$ has index 1 and nullity 0.*

On the other hand on Page space we have index and nullity strictly depending on the value of the parameter of the Berger sphere. In particular they have a strictly positive index which means the Berger sphere behaves contrary to the compact Kähler submanifolds in terms of index. Next we are able to find the following result inside the blow up of the complex projective plane.

Theorem 5.1. *The Index of the Berger sphere in the Page space is*

$$\text{Index}(\mathbb{S}_r^3) = \begin{cases} 5 & \text{for } r_1 \leq r \leq r_2 \\ 1 & \text{otherwise} \end{cases}$$

where r_1, r_2 are the roots of the following function

$$2f^{-1} + U^2 D^{-2} \sin^{-2} r - 3(1 + a^2)$$

in the interval $(0, \pi)$ and the nullity is

$$\text{Nullity}(\mathbb{S}_r^3) = \begin{cases} 4 & \text{for } r = r_1, r_2 \\ 0 & \text{otherwise} \end{cases}$$

At the end, we also give a criterion for instability of submanifolds.

Theorem 6.1. *Let $f : M \rightarrow \bar{M}$ be a totally geodesic immersion of a p -manifold into an Einstein n -manifold of positive scalar curvature. If M is a hypersurface or \bar{M} is of constant curvature then M is unstable.*

The paper is organized as follows. In section §2 we investigate the Jacobi operator for submanifolds in Einstein spaces, §3 we review and also give new computations on the spectrum of the Laplacian, in §4 and §5 we give applications to the complex projective space and its blow up. Finally in §6 we review the basic facts related to the stability of minimal submanifolds in a Riemannian manifold.

This is a sequel to our previous papers [KK15, KK18] and [KS21]. Further investigations in this direction takes place in the sequel [KKT24].

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2 The Jacobi Operator

Whenever we have an immersion $f : M \rightarrow \bar{M}$ of a p -manifold into an n -manifold, one can consider the Jacobi operator on the sections of the normal bundle,

$$J : C^\infty(NM) \rightarrow C^\infty(NM) \quad \text{defined by} \quad J = -\text{tr} \nabla^2 + \bar{R} - \tilde{S}.$$

Here, the third term $\tilde{S} = {}^T S \circ S$ is the composition of the transpose of the shape operator with itself considered as a map $S : NM \rightarrow \text{Sym}^2(TM)$. Another important term of the Jacobi operator is the so called *a partial Ricci operator* by [Sim68] acting on the normal bundle which is defined through

$$\bar{R}V := \sum_{i=1}^p (\bar{R}_{E_i V} E_i)^N \quad \text{for } E_1 \cdots E_p \in TM_m \quad \text{and } V \in NM_m$$

is non-zero unless the underlying manifold is Ricci-flat. If the ambient space is *Einstein* i.e. $Ric = \lambda g$, then this operator admits further decomposition. Moreover, if the immersed submanifold is totally geodesic the Jacobi operator has vanishing third term and thus admits the following simpler form.

Theorem 2.1. *Let $f : M \rightarrow \bar{M}$ be a totally geodesic immersion of a p -manifold into an Einstein n -manifold. The following are true.*

1. *The Jacobi operator of M is equal to $J = -\Delta_M - (s/n)I + Ric^\perp$.*
2. *If M is a hypersurface or \bar{M} is of constant curvature, then $J = -\Delta_M - (s/n)I$.*

Proof. In the tangent space we can pointwise complete to an orthonormal basis

$$\{E_1 \cdots E_p, E_{p+1}=V_1 \cdots E_n=V_{n-p}\}$$

of the ambient space \bar{M} . In order to understand the middle term of the Jacobi operator, we take the inner product of its image with an arbitrary normal basis element.

$$\begin{aligned} \langle \bar{R}V, V_j \rangle &= \langle \sum_{i=1}^p (\bar{R}_{E_i V} E_i)^N, V_j \rangle = \sum_{i=1}^p \langle \bar{R}_{E_i V} E_i, V_j \rangle \\ &= \sum_{i=1}^p \langle \bar{R}_{E_i V} E_i, V_j \rangle \pm \sum_{i=p+1}^n \langle \bar{R}_{E_i V} E_i, V_j \rangle \\ &= -Ric(V, V_j) - \sum_{i=1}^{n-p} \langle \bar{R}_{V_i V} V_i, V_j \rangle \\ &= \langle -Ric(V), V_j \rangle + \langle Ric^\perp(V), V_j \rangle, \end{aligned}$$

where

$$Ric^\perp : NM \rightarrow NM \text{ is defined by } V \longmapsto - \sum_{i=1}^{n-p} (\bar{R}_{V_i V} V_i)^N$$

is the Ricci curvature of the normal bundle. So that the partial Ricci operator term is $\bar{R} = -Ric + Ric^\perp$. In the Einstein case the constant is $\lambda = s/n$ which is a multiple of the constant scalar curvature.

In the case of $p = n - 1$ since the normal bundle is 1-dimensional, Ricci curvature of the normal bundle vanishes. So that

$$\bar{R} = -Ric = -\lambda I$$

then this operator acts as a multiple of the identity operator. \square

Along the way, we have also proved the following.

Proposition 2.2. *In the case of an immersion of a p -manifold into an Einstein n -manifold, the normal Ricci curvature vanishes if and only if the operator $\bar{R} = -(s/n)I$ i.e. a multiple of the identity operator.*

3 Spectrum of the Laplacian

In this section, we are going to review the spectrum of the Laplacian on some compact Riemannian manifolds. Some resources on the subject are [GHL04] and [BGM71] or more recent [Haj23]. We start with the unit round p -sphere S^p . To obtain a positive operator thus a positive spectrum,

$$-tr \nabla_{S^p}^2 P = k(k+p-1) P$$

where the Eigenfunctions in this case, constitute the space $E_k = H_k \mathbb{R}^{p+1}$ of the harmonic k -homogenous polynomials restricted from the Euclidean $(p+1)$ -space. Computation of the dimension of this space depends on the following observation[ABR01]. A k -homogenous polynomial in n -space has a harmonic part plus a part coming from the isomorphic image of lower degree polynomials \mathcal{P}_{k-2}^n under the multiplication map with radius squared. See the Table 1 for the details.

Table 1: *Spectrum of the Laplacian of the round p -sphere.*

k	0	1	2	3	...	k	...
<i>Spec:</i> λ_k	0	p	$2(p+1)$	$3(p+2)$...	$k(k+p-1)$...
<i>dim</i> E_{λ_k}	1	$p+1$	$(p+1)p/2-1$	$(p+1)p(p-1)/3!-(p+1)$...	$C_{k+p,p} - C_{k+p-2,p}$...

Next, we continue with the so-called *squashed sphere* or *the Berger sphere*. In general, one can define these metrics on the odd-dimensional spheres S^{2n+1} . The sphere is considered as the total space of the circle bundle on the complex projective space CP_n of first Chern class -1 . Then, it is furnished by the pullback of the Fubini-Study metric plus the fiber metric which makes the totally geodesic fibers equal in diameter. As a consequence of the fact that 3(and also 7)-dimensional sphere is parallelizable, on S^3 we have a simple expression for the Berger metric,

$$g_\epsilon := \sigma_1^2 + \sigma_2^2 + \epsilon^2 \sigma_3^2.$$

Here, ϵ is a positive constant and $\{\sigma_1, \sigma_2, \sigma_3\}$ are the standard left invariant 1-forms on the Lie group $SU_2 \approx S^3$. These can be expressed in terms of the variables based on the standard embedding of the 3-sphere into 4-space as follows,

$$\begin{aligned} r^2 \sigma_1 &= x_1 dx_0 - x_0 dx_1 + x_2 dx_3 - x_3 dx_2 \\ r^2 \sigma_2 &= x_2 dx_0 - x_0 dx_2 + x_3 dx_1 - x_1 dx_3 \\ r^2 \sigma_3 &= x_3 dx_0 - x_0 dx_3 + x_1 dx_2 - x_2 dx_1. \end{aligned}$$

[Tan79] computes the first nonzero eigenvalue for the Berger spheres in all odd dimensions which have the same volume as the round sphere of the dimension,

which we call unit volume for simplicity. Restricting his results to the dimension three one uses the family of unit volume Berger metrics on S^3 ,

$$g_B^t = t^{-1}g_1 + (t^2 - t^{-1})\sigma_3^2 \quad \text{for } t \in \mathbb{R}^+.$$

Tanno computes the smallest positive eigenvalue for this case as follows.

$$\lambda_1(g_B^t) = \begin{cases} 8t & \text{for } t \leq 1/\sqrt[3]{6} \\ t(2 + t^{-3}) & \text{for } t \geq 1/\sqrt[3]{6} \end{cases}$$

Normalizing the volume of the ϵ -Berger metric we get $g_\epsilon^u = \epsilon^{-2/3}g_\epsilon$ which corresponds to the $t = \epsilon^{2/3}$ case of Tanno's. So that we obtain the following identification,

$$g_\epsilon = \epsilon^{2/3}g_B^{\epsilon^{2/3}}.$$

To be able to apply Tanno, we need one more step. Laplace-Beltrami operator on functions in local coordinates reveals the fact that it anti-transforms under the dilation of the metric, i.e. $\Delta_{\lambda g}f = \lambda^{-1}\Delta_gf$. This result is true in the case of constant λ or if the space is a surface. To prove the latter assertion one needs to work out the general conformal scaling formula though. In the case that it holds, although the eigenfunctions stay the same, the eigenvalues are anti-multiplied by λ . Inserting the t value and anti-multiplying we finally arrive at the first eigenvalue.

$$\lambda_1(g_\epsilon) = \begin{cases} 8 & \text{for } \epsilon \leq 1/\sqrt{6} \\ 2 + 1/\epsilon^2 & \text{for } \epsilon \geq 1/\sqrt{6} \end{cases}$$

One can find further eigenvalues by using the following method. We need to sort eigenvalues in ascending order. To do that, for each step, we determine a minimum. To proceed further we take $m = 3$, as we are interested in three-dimensional Berger spheres we consider only this dimension. As in the proof above, the minimums come from the following three cases.

1. $tn(n+2) - t(1-t^{-3})n^2, \quad n \geq 1,$
2. $tk(k+2) - t(1-t^{-3}), \quad k \text{ is odd and } \geq 1,$
3. $tl(l+2), \quad l \text{ is even and } \geq 2.$

If we rearrange them we get the following curves.

1. $\gamma_n(t) = tn(2 + nt^{-3}), \quad n \geq 1,$
2. $\alpha_k(t) = tk(k^2 + 2k - 1 + t^{-3}), \quad k \text{ is odd and } \geq 1,$
3. $\beta_l(t) = tl(l+2), \quad l \text{ is even and } \geq 2.$

The first nonzero eigenvalue was found by comparing the minimums of those three and formed by splitting cases. In terms of the new notations, we can write the first eigenvalue as

$$\lambda_1(g(t)) = \begin{cases} \alpha_1(t) = \gamma_1(t) & \text{for } t^{-3} \leq 6 \\ \beta_2(t) & \text{for } t^{-3} \geq 6 \end{cases}.$$

To find $\lambda_2(g(t))$, we need to compare other possible curves, those are $\beta_2, \gamma_2, \alpha_3$, since α_3 is far bigger than others, after the comparisons we have

$$\lambda_2(g(t)) = \begin{cases} \gamma_2(t) & \text{for } t^{-3} \leq 1 \\ \beta_2(t) & \text{for } t^{-3} \geq 1 \end{cases}.$$

If we proceed further we realize that we first start with $\beta_2(t)$, then pass to possible α_k which is α_1 , then pass possible γ_n , which is γ_1 but since they are equal we take both at the same time but this never be the case again. Then for the next eigenvalue, we start with β_2 , then try to pass α_3 but $\alpha_3 > \beta_2$ hence we move with γ_2 . For the next eigenvalue, again the same logic: start with β_2 then move with γ_3 . We proceed with this until no γ under β_2 is left. Whenever there is no γ left under β_2 we take β_2 for all t as eigenvalue. The next step moves with β_4 , then we try α_3 as next, and other γ 's. We move with β_4 and α_3 until no γ left under them, then we pass two cased eigenvalue β_4 and α_3 , then we pass to the cases β_4 and other γ 's lie between α_3 and β_4 . Then we pass β_4 as eigenvalue itself. This process continues like this. Roughly speaking we start our eigenvalue with β_l as the first case then take α_{l-1} as the second case, and then take suitable γ as the third case. We increase the index of γ until they become greater than α_{l-1} . Whenever they do, we take the next eigenvalue in the form

$$\lambda_p(g(t)) = \begin{cases} \alpha_{l-1}(t) & \text{for } t^{-3} \leq A \\ \beta_l(t) & \text{for } t^{-3} \geq A \end{cases}.$$

The next few eigenvalues will have form

$$\lambda_q(g(t)) = \begin{cases} \gamma_n(t) & \text{for } t^{-3} \leq A \\ \beta_l(t) & \text{for } t^{-3} \geq A \end{cases}$$

until γ_n 's start to become greater then β_l . Whenever they do we select the next eigenvalue as β_l itself. Then we increase the index of β and move in this algorithm. By using this algorithm, we obtained the first eleven eigenvalues as follows.

$$\lambda_1(g(t)) = \begin{cases} t(2 + t^{-3}) & \text{for } t^{-3} \leq 6 \\ 8t & \text{for } t^{-3} \geq 6 \end{cases}'$$

$$\lambda_2(g(t)) = \begin{cases} 2t(2 + 2t^{-3}) & \text{for } t^{-3} \leq 1 \\ 8t & \text{for } t^{-3} \geq 1 \end{cases}'$$

$$\lambda_3(g(t)) = \begin{cases} 3t(2 + 3t^{-3}) & \text{for } t^{-3} \leq \frac{2}{9}, \\ 8t & \text{for } t^{-3} \geq \frac{2}{9}, \end{cases}$$

$$\lambda_4(g(t)) = 8t$$

$$\lambda_5(g(t)) = \begin{cases} 4t(2 + 4t^{-3}) & \text{for } t^{-3} \leq \frac{2}{5}, \\ t(14 + t^{-3}) & \text{for } \frac{2}{5} \leq t^{-3} \leq 10, \\ 24t & \text{for } t^{-3} \geq 10 \end{cases},$$

$$\lambda_6(g(t)) = \begin{cases} 5t(2 + 5t^{-3}) & \text{for } t^{-3} \leq \frac{1}{6}, \\ t(14 + t^{-3}) & \text{for } \frac{1}{6} \leq t^{-3} \leq 10, \\ 24t & \text{for } t^{-3} \geq 10 \end{cases},$$

$$\lambda_7(g(t)) = \begin{cases} 6t(2 + 6t^{-3}) & \text{for } t^{-3} \leq \frac{2}{35}, \\ t(14 + t^{-3}) & \text{for } \frac{2}{35} \leq t^{-3} \leq 10, \\ 24t & \text{for } t^{-3} \geq 10 \end{cases},$$

$$\lambda_8(g(t)) = \begin{cases} t(14 + t^{-3}) & \text{for } t^{-3} \leq 10 \\ 24t & \text{for } t^{-3} \geq 10 \end{cases},$$

$$\lambda_9(g(t)) = \begin{cases} 7t(2 + 7t^{-3}) & \text{for } t^{-3} \leq \frac{10}{49}, \\ 24t & \text{for } t^{-3} \geq \frac{10}{49}, \end{cases}$$

$$\lambda_{10}(g(t)) = \begin{cases} 8t(2 + 8t^{-3}) & \text{for } t^{-3} \leq \frac{1}{8}, \\ 24t & \text{for } t^{-3} \geq \frac{1}{8}, \end{cases}$$

$$\lambda_{11}(g(t)) = \begin{cases} 9t(2 + 9t^{-3}) & \text{for } t^{-3} \leq \frac{2}{27}, \\ 24t & \text{for } t^{-3} \geq \frac{2}{27}. \end{cases}$$

Remark 3.1. Other spectrum computations for Riemannian manifolds include complex projective spaces and tori. See also [BBB82].

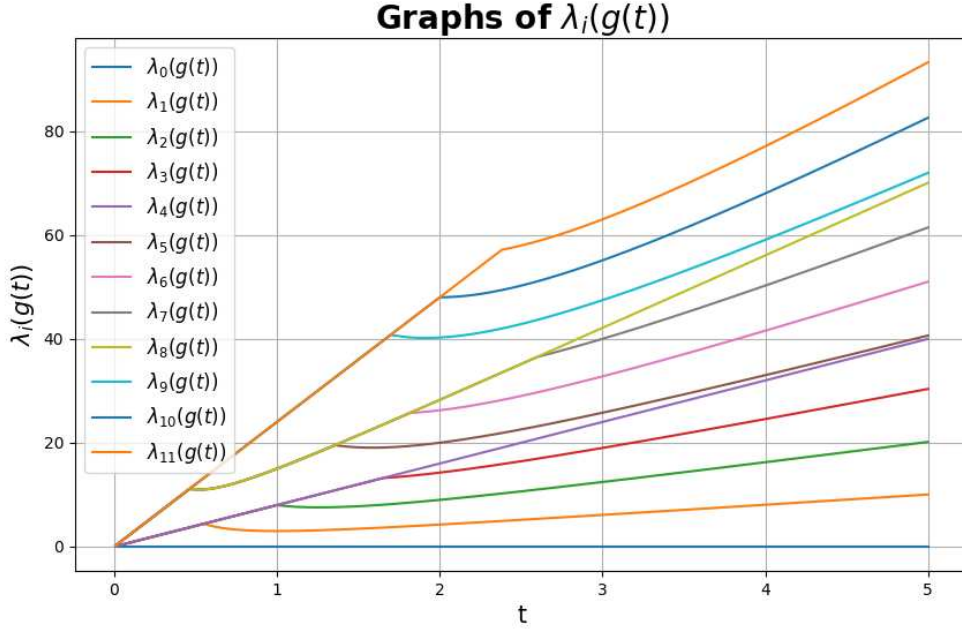


Figure 1: Eigenvalues for the Berger sphere Laplacians

4 Index of minimal submanifolds of the complex projective space

In this section we apply the machinery in the previous sections to study the hypersurfaces of the complex projective plane. We are going to review some facts about the minimal submanifolds of the complex projective spaces. For simplicity we work on dimension two. From the Hermitian metric

$$G = (1 + |z_1|^2 + |z_2|^2)^{-2} \begin{bmatrix} 1 + |z_2|^2 & -\bar{z}_1 z_2 \\ -z_1 \bar{z}_2 & 1 + |z_1|^2 \end{bmatrix}$$

one gets the real part as,

$$(1 + x_1^2 + y_1^2 + x_2^2 + y_2^2) g_{FS} = (1 + x_2^2 + y_2^2) \{dx_1^2 + dy_1^2\} + (1 + x_1^2 + y_1^2) \{dx_2^2 + dy_2^2\} \\ - (x_1 x_2 + y_1 y_2) \{dx_1 \otimes dx_2 + dx_2 \otimes dx_1 + dy_1 \otimes dy_2 + dy_2 \otimes dy_1\}.$$

An orthogonal coframe for the Fubini-Study metric is as follows.

$$\{e^1, e^2, e^3, e^4\} := \left\{ dx_1, dy_1, dx_2 - \frac{x_1 y_2 - x_2 y_1}{1 + x_1^2 + y_1^2} dx_1, dy_2 - \frac{x_1 y_2 - x_2 y_1}{1 + x_1^2 + y_1^2} dy_1 \right\}.$$

When we have a Kähler manifold, complex submanifolds are homologically volume minimizing among other submanifolds with the same boundary by the result

of [Fed65] who seems to be the first to show that. Thus it makes the index equal to zero for complex submanifolds. Later on, [Sim68] was able to show that the Jacobi fields are holomorphic sections of the normal bundle thus able to express the nullity by the number of these sections. For a complex surface S with a smooth, connected complex curve C in it, the adjunction formula [GS99] reads as follows.

$$2g(C) - 2 = [C]^2 - c_1(S)[C]$$

If we consider $g = 0$ in $\mathbb{C}P_2$ the only possibilities for the degree of the embedding is $d = 1, 2$. In these cases the self intersections become $[C]^2 = 1, 4$.

Theorem 4.1. *The degree = 1, 2 embeddings of $\mathbb{C}P_1$ in $\mathbb{C}P_2$ has nullity = 1, 4. The linear embedding of $\mathbb{C}P_{n-1}$ in $\mathbb{C}P_n$ is totally geodesic and has nullity = 1. All of these embeddings have index zero.*

On the other hand, for the real hypersurface case, as an application of the lemmas and theorems in the previous sections, we obtain the following result.

Theorem 4.2. *The Berger sphere in $\mathbb{C}P_2$ has index 1 and nullity 0.*

Proof. We prove this by finding the eigenvalues of Berger sphere in $\mathbb{C}P_2$, then by using Theorem 2.1, evaluate the eigenvalues of the Jacobi operator. Let us start with zero eigenvalue $\lambda_0 = 0$, by using Theorem 2.1, we can see that the first eigenvalue of the Jacobi operator is $-\frac{3}{2}$, and the dimension is 1 by [Tan79]. Now we continue with nonzero eigenvalues. We apply Tanno's idea to our Berger sphere, note that Tanno has parametrized the metric as

$$t^{-1}g + (t^{2n} - t^{-1})\eta \otimes \eta$$

here if we consider S^3 , we have σ_3 produced from a unit Killing vector field because the coefficient of the metric is constant, hence all $\sigma_1, \sigma_2, \sigma_3$ are Killing, hence we can use it as η . Then metric becomes

$$t^{-1}(\sigma_1^2 + \sigma_2^2) + t^2\sigma_3^2. \quad (4.1)$$

The problem is, our Berger sphere metric does not come from a unit sphere thus we cannot calculate the first eigenvalue for that by using directly Tanno's idea. Instead, we rescale our metric such that the resulting metric's volume is the same as S^3 's volume. Remember that the metric for Berger sphere in $\mathbb{C}P_2$ is

$$g_{BS} = f(\sigma_1^2 + \sigma_2^2) + r^2V\sigma_3^2$$

where r is fixed real number and $f = r^2/(1 + r^2)$, $V = (1 + r^2)^{-2}$. To rescale this metric, we first calculate the volume.

$$\sqrt{\det(g_{BS})} = \frac{r^3}{(1 + r^2)^2}$$

hence

$$\text{Vol}(g_{\text{BS}}) = \frac{r^3}{(1+r^2)^2} \text{Vol}(\bar{g})$$

where \bar{g} is the metric for the Berger sphere in unit sphere \mathbb{S}^3 with the volume same as the volume of \mathbb{S}^3 . Hence we have a scaling factor

$$\mu = \frac{r^2}{(1+r^2)^{4/3}}$$

as the scaling of metric by μ impacts volume by $\mu^{n/2}$ where n is the dimension. At the end of the day, we have the relation

$$g_{\text{BS}} = \frac{r^2}{(1+r^2)^{4/3}} \bar{g}$$

where \bar{g} is the Berger sphere induced from unit sphere \mathbb{S}^3 and has the same volume with \mathbb{S}^3 . This metric can be explicitly given as

$$\bar{g} = (1+r^2)^{1/3}(\sigma_1^2 + \sigma_2^2) + (1+r^2)^{-2/3}\sigma_3^2.$$

To receive Tanno's idea we need to agree with it in notation. If we compare the metrics above and in the equation 4.1, we have the relationship between t and r as $t = (1+r^2)^{-1/3}$. Hence we can give the first eigenvalue for the Laplacian for \bar{g} as

$$\bar{\lambda}_1(r) = \begin{cases} (3+r^2)(1+r^2)^{-1/3} & \text{for } r \leq \sqrt{5} \\ 8(1+r^2)^{-1/3} & \text{for } r \geq \sqrt{5} \end{cases}$$

to calculate the index of the Jacobi operator, we need to calculate the first eigenvalue for g_{BS} . To do that we need to use the fact that the Laplacians of the metrics scaled by μ are related with the Laplacian of the original metric by $1/\mu$. Hence we need to multiply the first eigenvalue for the Laplacian of \bar{g} by $r^{-2}(1+r^2)^{4/3}$. Therefore the first eigenvalue for g_{BS} can be given as

$$\lambda_1(r) = \begin{cases} \frac{3+r^2}{r^2}(1+r^2) & \text{for } r \leq \sqrt{5} \\ 8\frac{1+r^2}{r^2} & \text{for } r \geq \sqrt{5} \end{cases}.$$

For the index of the Jacobi operator, by Theorem 2.1, we need to find the dimensions of eigenspaces of negative eigenvalues of $J = \Delta_{\text{BS}} - 3/2 I$. The first eigenvalue for J is

$$\lambda_1(r) - 3/2 = \begin{cases} \frac{3+r^2}{r^2}(1+r^2) - \frac{3}{2} & \text{for } r \leq \sqrt{5} \\ 8\frac{1+r^2}{r^2} - \frac{3}{2} & \text{for } r \geq \sqrt{5} \end{cases}$$

which is strictly positive for all values of $r > 0$. Hence the index is 1 and since there is no zero eigenvalue, the nullity of the Berger sphere metric g_{BS} is zero. \square

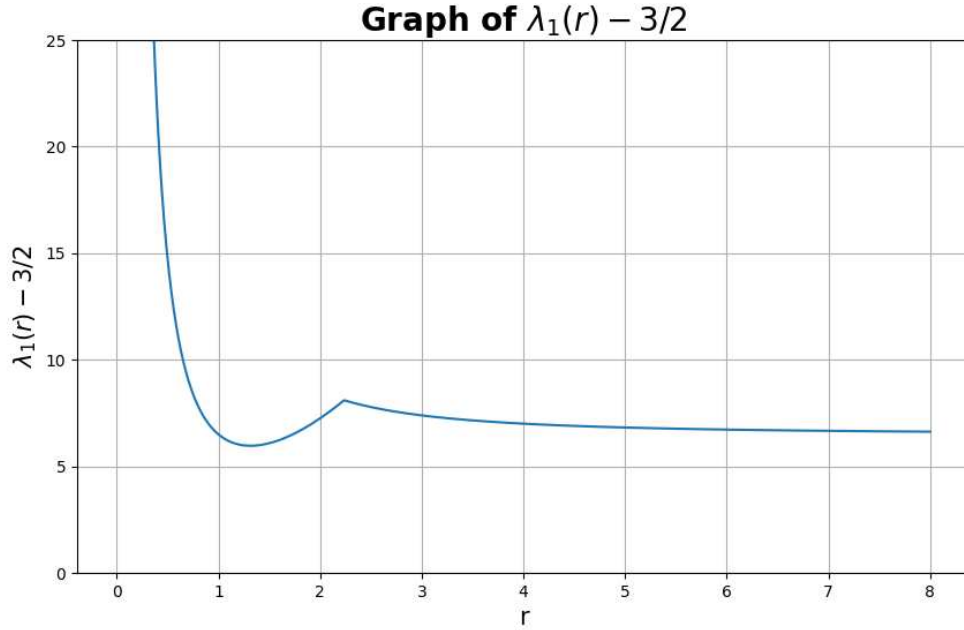


Figure 2: The first eigenvalue of the Jacobi operator

5 Submanifolds of the Page Space

In this section, similar to the index section for $\mathbb{C}P_2$, we use the Theorem 2.1, and Tanno's idea we reach out to the index of the Berger sphere in the Page space. First, the Berger sphere in Page space can be given by fixing the r coordinate in Page metric. The resulting metric is given by

$$g_{BS} = f(\sigma_1^2 + \sigma_2^2) + \frac{C \sin^2 r}{V} \sigma_3^2$$

which is totally geodesic in Page space by [KS21]. Here, the possible values for r is the interval $(0, \pi)$.

Theorem 5.1. *The Index of the Berger sphere in the Page space is*

$$\text{Index}(\mathbb{S}_r^3) = \begin{cases} 5 & \text{for } r_1 \leq r \leq r_2 \\ 1 & \text{otherwise} \end{cases}$$

where r_1, r_2 are the roots of the following function

$$2f^{-1} + U^2 D^{-2} \sin^{-2} r - 3(1 + a^2)$$

in the interval $(0, \pi)$ and the nullity is

$$\text{Nullity}(\mathbb{S}_r^3) = \begin{cases} 4 & \text{for } r = r_1, r_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. As in the proof for $\mathbb{C}\mathbb{P}_2$ we first normalize the Berger sphere. The volume for the Berger sphere is

$$\text{Vol}(g_{BS}) = \frac{fD \sin r}{U} \text{Vol}(\bar{g})$$

where \bar{g} is the metric for the Berger sphere in the unit sphere \mathbb{S}^3 with the volume same as the volume of \mathbb{S}^3 . Hence we have a scaling factor

$$\mu = \left(\frac{fD \sin r}{U} \right)^{2/3}$$

Hence we get

$$g_{BS} = \left(\frac{fD \sin r}{U} \right)^{2/3} \bar{g}$$

We can give the metric \bar{g} explicitly as

$$\bar{g} = \left(\frac{D \sin r}{U} \right)^{-2/3} f^{1/3} (\sigma_1^2 + \sigma_2^2) + \left(\frac{D \sin r}{U} \right)^{4/3} f^{-2/3} \sigma_3^2.$$

If we compare the metrics above and in the equation 4.1, we have the relationship between t and r as $t = U^{-2/3} (D \sin r)^{2/3} f^{-1/3}$. Hence we can give the first nonzero eigenvalue for the Laplacian for \bar{g} as

$$\bar{\lambda}_1(r) = \begin{cases} \left(2 + \left(\frac{D \sin r}{U} \right)^{-2} f \right) \left(\frac{D \sin r}{U} \right)^{2/3} f^{-1/3} & \text{for } \left(\frac{D \sin r}{U} \right)^{-2} f \leq \sqrt{6} \\ 8 \left(\frac{D \sin r}{U} \right)^{2/3} f^{-1/3} & \text{for } \left(\frac{D \sin r}{U} \right)^{-2} f \geq \sqrt{6} \end{cases}$$

to calculate the index of the Jacobi operator, we need to calculate the first nonzero eigenvalue for g_{BS} . To do that we need to multiply the first eigenvalue for the Laplacian of \bar{g} by $\mu^{-1} = U^{2/3} (fD \sin r)^{-2/3}$. Now we have the first eigenvalue for g_{BS} . For the index of the Jacobi operator, by Theorem 2.1, we need to find the dimensions of eigenspaces of negative eigenvalues of $J = \Delta_{BS} - s/4 I$, by [KS21], we know the scalar curvature for Page space is $12(1 + a^2) \simeq 12.952$. If we apply those, we see that the first eigenvalue for the Jacobi operator is $\lambda_0(r) - 3(1 + a^2)$, and the second eigenvalue is

$$\lambda_1(r) - 3(1 + a^2) = \begin{cases} \left(2 + \left(\frac{D \sin r}{U} \right)^{-2} f \right) f^{-1} - 3(1 + a^2) & \text{for } \left(\frac{D \sin r}{U} \right)^{-2} f \leq \sqrt{6} \\ 8f^{-1} - 3(1 + a^2) & \text{for } \left(\frac{D \sin r}{U} \right)^{-2} f \geq \sqrt{6} \end{cases}$$

If we calculate the roots for $\lambda_1(r) - 3(1 + a^2)$ by using bisector method with tolerance 10^{-6} , we get the roots $r_1 \simeq 0.7032761573791504$ and $r_2 = 2.4383171081542976$. Hence the index and nullity for the interval $(0, r_1)$ and (r_2, π) are 1 and zero, respectively as the dimension of the eigenspace for $\lambda_0(r)$ is 1, however for (r_1, r_2) the

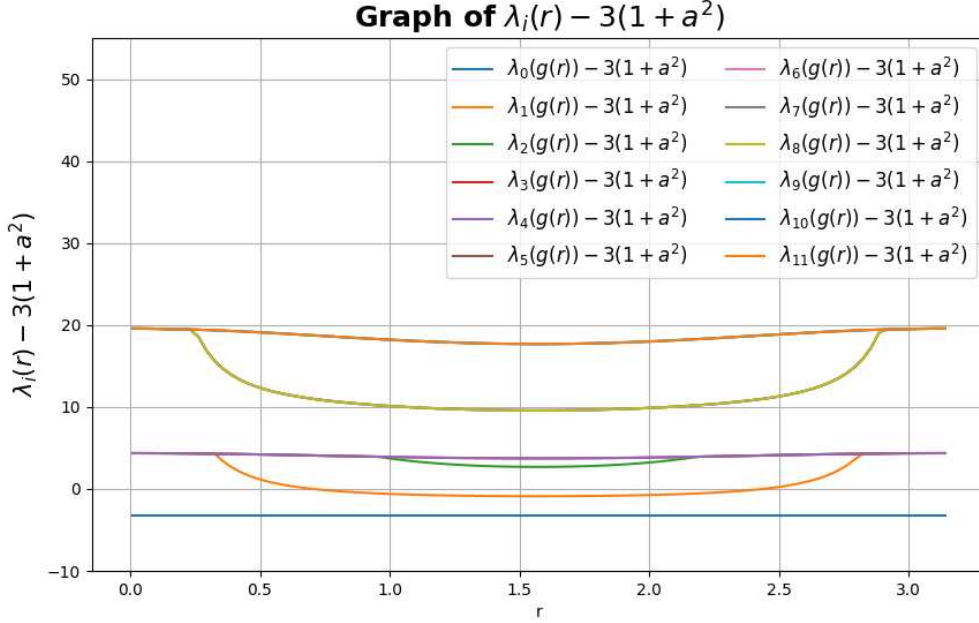


Figure 3: Eigenvalues of the Jacobi operator for the Berger spheres in Page space

index is greater than the other two intervals. If we sketch the other eigenvalues for the Jacobi operator we get the following graph. Some eigenvalues are overlapped due to the interval for the cases. One can see that nullity is nonzero only for two values of r , and index is 5 for (r_1, r_2) as the dimension for the eigenspace of $\lambda_1(r)$ is $m + 1 = 4$ by [Tan79]. Hence the index on this interval is 5, on the outer part is 1. \square

6 Stability of minimal submanifolds

The second variational formula for the volume in terms of compactly supported normal variations is computed to be [Li12],

$$\frac{d^2}{dt^2}V(N_t)|_{t=0} = \int_N \left\{ - \sum_{i,j} \langle T, II_{i,j} \rangle^2 - \sum_{i=1}^n \langle \mathcal{R}_{e_i T} T, e_i \rangle + \langle \nabla_T^n T, H \rangle + \sum_{i=1}^n \sum_{\alpha=n+1}^m \langle \nabla_{e_i} T, e_\alpha \rangle + \langle T, H \rangle \right\}.$$

Since complex submanifolds of Kähler manifolds are also Kähler, by the theorem of [Sim68] complex submanifolds of the complex projective space has zero index. There exist no negative eigenvalue for the Jacobi operator. So that there are no directions in the normal bundle to deform are to make area smaller. So that deformations of them are initially area increasing which we call *stable*. This is contrary to the case of the round spheres living inside round spheres. The Theorem 1.2 of

Simons that we have cited in the introduction implies that all minimal submanifolds of the round sphere is unstable. A criterion can be given for instability using results from section §2.

Theorem 6.1. *Let $f : M \rightarrow \bar{M}$ be a totally geodesic immersion of a p -manifold into an Einstein n -manifold of positive scalar curvature. If M is a hypersurface or \bar{M} is of constant curvature then M is unstable.*

Proof. This falls into the second case of the Theorem 2.1. Since the Laplacian has nonnegative (and discrete) eigenvalues, smallest one is zero with multiplicity one, the dimension of the constant functions. This yields a negative eigenvalue for the Jacobi operator after shifting by the scalar curvature. Consequently the index is greater than or equal to one. \square

As an application of this corollary, we can say that the totally geodesic hypersurfaces of $\mathbb{C}P_2$ and the Page space are unstable, in particular the Berger spheres that we are interested in this paper. Our direct computation in section §4 and §5 is an alternative verification of this result. The minimal submanifolds in this class behave more like the ones in spheres in terms of this respect rather than complex submanifolds.

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